2/1/ü4

NASA TECHNICAL TRANSLATION NASA TT F-15,066

A STUDY OF POLAR MOTION

A. Terpos

(NASA-TT-F-15066) A STUDY OF POLAR MOTION M.S. Thesis (Kanner (Leo) Associates) 249 p HC $14.50 CSCL 08E

N73-33334

Unclas
G3/13 19810

Translation of Peri tēs kinēseos tou polou tēs gēs, Dissertation for the Master's Degree, Division of Rural and Surveying Engineering, University of Thessaloniki Polytechnic School, 1971, 287 pp.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 OCTOBER 1973

Table of Contents

*This section is not included in the work, although it is listed in the table of contents.

Page

Page

A STUDY OF POLAR MOTION

Antonios Terpos
University of Thessaloniki Polytechnic School
Division of Rural and Surveying Engineering

1. Introduction (Historical) /6*

The study of polar motion begins actually in the year 1765, when Euler published a classic paper in which he examines what the laws describing Earth's rotational motion would be if the Earth were a rigid body having the shape of an ellipsoid. Euler demonstrated that, under these conditions, there are two possibilities:

1. If the inertial axis of the Earth coincides initially with the principal inertial axis, i.e., the rotational axis of the corresponding ellipsoid, then the Earth will keep rotating around its principal rotational axis. Therefore, in this case, the rotational axis of the Earth remains at a fixed position, with respect to its mass, and the points of intersection of this axis with the surface of the Earth, i.e., the geographical poles of the Earth, will be fixed points on the surface of the Earth.

2. If the rotational axis of the Earth does not initially coincide with its principal inertial axis, then the rotational axis will move continuously around the principal inertial axis of the Earth, describing a cone by revolution, a cone which has as a vertex the center of mass of the Earth. The position of the rotational axis of the Earth, at an arbitrary moment, is called <u>instantaneous rotational axis</u> in this case, and the points of intersection of this axis with the surface of our planet are called <u>instantaneous geographical poles</u>, or simply <u>instantaneous poles</u> of the Earth at the moment under consideration.

Therefore, each one of the instantaneous poles of the Earth describes, in this case, the circumference of a small circle on the surface of the Earth, the center of whose circumference coincides with the point at which the principal rotational axis of the Earth intersects the corresponding hemisphere of the Earth.

This phenomenon is known as "polar motion." Euler found that the period of the polar motion is equal to 304 mean solar days. Euler showed that the radius of the circular trajectile of the pole can not be computed theoretically, but it has to be determined using observations. Following Euler's introduction, namely that the Earth is subjected to a <u>free nutation</u> with period /7

*Numbers in the margin indicate pagination in the foreign text.

A/(C+A) sidereal days, Peter in 1841, Bessel in 1842, and Maxwell in 1851 investigated the latitude changes with a [illegible] period of 10 months. The results of these investigations, which are proved with difficulty, showed changes on the order of less than 0".1 and of doubtful importance. However, Lord Kelvin suggested that the results can be of extreme importance. His opinion was based on geophysical considerations, and he had calculated that the displacement in the aerial mass would cause a wobble on the order of 0".05 to 0".5. In Kelvin's inquiry, Newcomb analyzed (studied) the latitude of Washington during the years 1862-1865, for changes of 10 months, and he got a difference in value of about 0".05 ≠ 0".03. This result was announced by Kelvin during the presidential ceremony at the British Association as a proof of free nutation.

When the phenomenon was examined, it was proven that Kelvin was right as far as the event went; he was, however, mistaken in computing the period. The exact solution was found, perhaps characteristically, during an investigation of a completely different kind, and before an opinion had been formed about change frequency.

In 1884, Künster, in Berlin, started a series of measurements in order to determine the constant of aberration, using small differences in siderial zenith distances, according to a method discovered by Tolcott (U.S. Association of Engineers). /8

He was surprised when he found a change in the above constant, an almost annual one. Having personally examined all possible values for errors, he was led to conclude that this was due to a change in latitude on the order of 0".2. The result was announced at the convention of 1888 in Salzburg, and the International Geodetic Association immediately became interested in the matter. A rigorous investigation was carried out in 1821 [sic]. Measurements of the latitude were taken simultaneously in Waikiki and in Berlin. The two stations are separated almost exactly by 180° in meridian longitude, i.e., they lie upon the same meridian. If, however, the rotational axis is continuously changing position within the mass of the Earth, then the astronomical geographic coordinates of different places on the Earth's surface must also be changing continuously. Indeed, the expected result was achieved with remarkable success. The change indicated was on the order of 0".5, and according to Professor Förster of Berlin, Kelvin's prediction of 1876 had been completely verified, and even the "abnormal movements of the Earth's axis, on the order of 0".5 can be attributed to the temporary changes of sea level, due to meteorological causes."

During that time, S.C. Chandler, in Cambridge, had begun a complete analysis of changes, trying to discover the changes in latitude since the era of Bradley, i.e., about 200 years earlier, and to show that many disagreements are due to /9 latitude changes. One of the first announcements Chandler made was that the observations showed a time term with a period of 428 days, i.e. about 40% longer than the classical value of Euler. This result was not the one expected, and it caused doubts as to the validity of the observations. However, Newcomb showed, 1 year later, that the retreat of the Earth and the oceans could give exactly such an increase in the period of 10 to 14 months (Newcomb, 1892). He attributed the change to the easily moving oceans, the remaining section to an elastic retreat of the Earth. With this remark, Newcomb suggested that latitude observations can show, in one of the best ways, the determination of rigidity (plasticity) of the Earth. Later, Chandler discussed the probability of existence of an eternal term in the latitude change; however, he could not discover such a change. Chandler also mentioned that he had determined some rather important variations with a nonannual period.

For the better organization and coordination of research relative to the phenomenon of polar motion, the "International Latitude Service" (I.L.S.) was established in 1898, which was renamed the "International Polar Motion Service" in 1963. Important observations of this service on polar motion are made in observatories which are known as international latitude observatories. In order that we be able to use the same stars, /10
the observatories chosen were all on the same parallel, 39°0.8' and distributed as symmetrically as possible, according to geographical longitude. Among the observatories mentioned here, only those of Muzusawa and Ukiah have operated without interruption until the present.

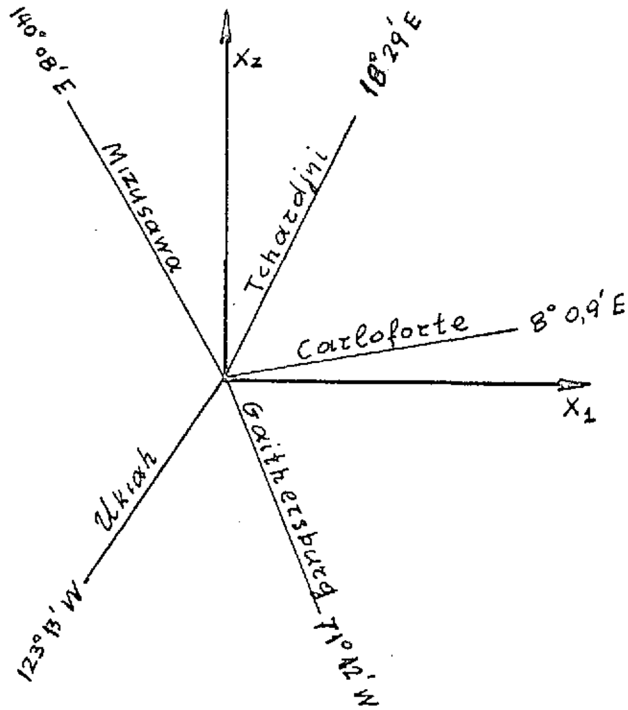
Unfortunately, the program of observations and reductions was not continuous. We distinguish three periods:

1. The German, 1900-1922.7.

2. The Japanese (Kimura) 1922.7-1935.

3. The Italian, 1935--.

At the international latitude observatories, a determination of the astronomical latitude of the place was performed every clear night with the aid of an "optical zenith telescope" by the method of Horrebow-Talcott. In 1912, the "floating zenith telescope" of Cookson was introduced. In 1915, the photographic zenith tube of Ross (PZT) was introduced in Greenwich. This was introduced to Washington (U.S. Naval Observatory) and later to other observatories, and finally to Greenwich and Mizusawa. It is hoped that all the I.L.S. observatories will be supplied with a PZT tube. Besides the PZT, the "impersonal



astrolabe" of Danjon, of the same accuracy as the PZT, is also widely used. In order to guarantee a greater accuracy, it is necessary that all the observations now performed in the I.L.S. observatories be done in Mizusawa, where the officies of the Polar Motion Service have been located since 1962.

2. Precession, Nutation and Wobble /11

2.1. Wobble and Precession

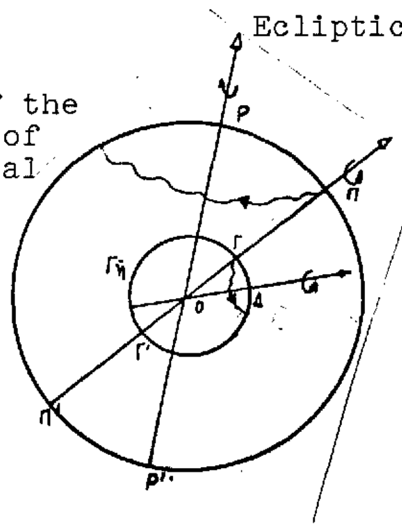
Let us assume that we take a picture of the stars with a camera directed vertically upward, i.e., with direction opposite the vector denoting the direction of the gravitational field of the Earth at that point. On these photographs we observe that a star describes traces which seem to belong to concentric circles. There are two positions on the Earth which almost constitute the center of these concentric circles (circumferences). We shall call these points poles of rotation. We define the two points of the celestial sphere with respect to which a star has no daily motion as instantaneous poles vertically above the poles of rotation. The axis passing through the poles of rotation and extending from one instantaneous pole to the other is called axis of rotation (instantaneous).

If we note the positions of the poles of rotation within a few feet (0.305 m) from their previous position, if the comparison of the position is made after the lapse of a year, we will observe that the poles have moved along an abnormal elliptic trajectile of a mean diameter of 6.10 m.

We define a fixed position of the pole leading to a base near the center of the ellipse. This base defines the reference pole. Finally, the axis extending from the reference pole to the center of the Earth is called the reference axis.

It is obvious from the above definitions that the reference pole assumes a mean position among the consecutive positions of the pole.

Therefore, the pole of rotation will describe a trajectile around the reference pole. With respect to an observation on a fixed star, the axis of rotation (instantaneous) remains fixed, while the reference pole (which is determined for a daily motion on the average) describes a trajectile around the pole of rotation. /12



Now a distinction must be made among the following phenomena.

The instantaneous rotational axis describes within the Earth a trajectile around the mean position of the reference axis. It also describes a trajectile on the celestial sphere around its North Pole in 25,796 years. The first phenomenon corresponds to the polar motion and has various causes. However, the trajectile of the instantaneous pole around the North Pole of the celestial sphere is due to the phenomenon of precession of the equinoxes and nutation of the axis of the celestial sphere.

In these two phenomena we observe the following. The change of the instantaneous rotational axis in space causes a continuous change of the coordinates of the stars on the celestial sphere, while the astronomical coordinates of various places on the surface of the Earth remain fixed. Conversely, because of the motion of the instantaneous rotational axis of the Earth within the Earth, the astronomical coordinates of various places of the Earth and the astronomical azimuths of various targets in these places continuously change, while the coordinates of the stars on the celestial sphere remain fixed. Notice that the changes of the astronomical coordinates and of the astronomical /13 azimuths which are caused by polar motion offer special interest, because they have important consequences on the astronomical determinations, the absolute meridian observations, and exact time measurement.

In summary, we conclude from the above that the instantaneous rotational axis of the Earth undergoes changes of orientation in space. These changes are connected with precession and nutation. There are, besides these changes, changes of small period in the position of the instantaneous pole also, or forced nutations. It should be noted that these changes of instantaneous pole caused by change in orientation of the rotational axis of the Earth are completely different from the wobbles of the Earth relative to the axis and have different causes. In other words, a distinction of two kinds of motion can be made:

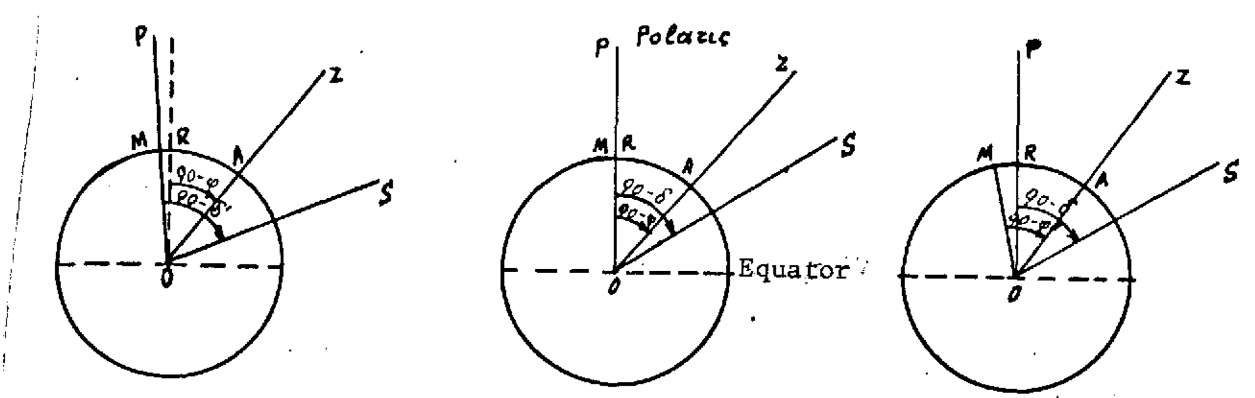
1. Change of the instantaneous axis in space because of precession and nutation. A periodic phenomenon.

2. Change of the position of the instantaneous pole on the Earth. Resultant position due to a series of phenomena of various small periods or even anomalies (wobble).

It would be disadvantageous if the discovery and study of the phenomena of precession and wobble could only be made from observations of the poles. Actually, the measurement of the proper angle can be made in every value of the latitude.

The angles which we determine are the ones which give the declination of a star and the latitude of the place of observation.

The following figure shows the position for the measurement of the angles for the ideal case of precession or forced nutation (left), and wobble (right) (i.e., angle measured from the center /14
with respect to the plane of the equator).



Let S be a star; then ZA gives the direction of the gravitational field (Z = zenith), and A is the fixed position on the Earth.

The latitude ϕ is defined by the relation 90° - $\hat{POZ}$ (co-latitude), where $\hat{POZ}$ is the angle between the instantaneous rotational axis (or the instantaneous pole) and OZ (or the zenith).

Let us consider the left-hand figure. The instantaneous pole has been moved away from polaris. As a consequence, we have a change in declination but not necessarily a change in latitude. This is the case of precession.

Let us now consider the figure on the right. The reference pole M has been displaced to the left from the rotational pole R. The declination remains the same; however, the latitude has changed. This is the case of wobble.

Hence it follows that the change in declination (δ) determines precession, while wobble is determined by the change in latitude.

2.2. Cause of Precession and of Forced Nutation

The changes of rotational axis of the Earth in space are mainly caused by the attraction of the Moon and the Sun in the equatorial swelling of the Earth. This phenomenon would not take place had the Earth not been a sphere or had the equatorial plane coincided with the elliptical plane of the orbit of the Sun and with the plane of the orbit of the Moon. Originally, however, an angle of 23°27' was formed with the elliptical plane of the Sun and an angle 23°27' ± (5°9')k (k = real in the interval [0,1]) with the Moon. /15

Had the earth not been rotated, the result of such an attraction would be the coincidence of the orbit and the equatorial plane. However, because of the rotation, we will have the gyroscopic phenomenon of rotation, i.e. the instantaneous rotational axis of the Earth will describe a clockwise orbit within 25,796 years. The obliquity of the ecliptic of the orbit remains close to 23°27'. This phenomenon is known as precession of equinox. If we call Δ, A, C the moment of inertia with respect to the principal axes of the Earth where C is the greatest value of the moment of inertia from the observed precession and the mass of the Moon, we can compute the constant of precession.

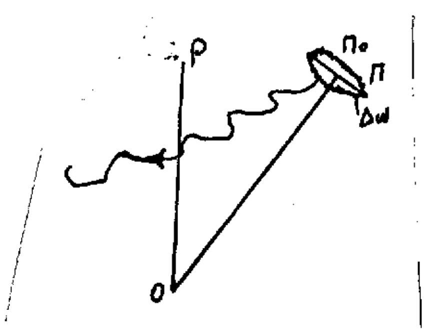
$$H = \frac{C - A}{A} = 0.00327293 \pm 0.00000075$$

Besides the Sun and the Moon, the other planets also apply a small attraction, giving a small precession. The combination of the orbits of the Sun and Moon is connected with wobbles of small period around 18 2/3 nutations of the Moon. The small periods imply originally a motion of the instantaneous pole to and from the poles of the ecliptic, an inclination which is called nutation (forced nutation ≠ Eulerian free nutation).

We conclude from the above that because of the noncoincidence of the equatorial swelling of the Earth with the orbits of the Sun and Moon, we have the phenomenon of precession. For the same reason, we also have the phenomenon of nutation (or forced nutation). Especially for nutation, because of the change of the inclination of the plane of the Moon orbit with respect to /16
the equatorial plane having a period of 18 2/3 years, we will finally have a composite curve described because of the motion of the instantaneous axis sweeping the surface of a cone within 25,796 years, and because of a motion of it sweeping the surface of an elliptic cone at the same time within 18 2/3 years.

(That is, precession is a result of the Moon and the Sun; nutation is a result mainly of the change of the Moon's orbit and less because of the attraction of the Moon and the Sun.)

Corresponding figure.



The equations giving precession and nutation are:

$$(\gamma\gamma_0) = \alpha(t - t_0) + \Delta y$$

$$\omega = \omega_0 + \Delta\omega$$

where γ_0 is the position of the vernal point γ at a certain moment and γ is its position after a time period t, ω_0 is the mean value of the obliquity of the ecliptic, a = 50°.4 and

$$\Delta y = p \sin 2\lambda_S + q \sin 2\lambda_M + b \sin\Omega + c \sin\Omega + \dots$$

$$\Delta\omega = p_1 \cos 2\lambda_S + q_1 \cos 2\lambda_M + b_1 \cos\Omega + c_1 \cos\Omega + \dots$$

where λ_S = geocentric longitude of the Sun; b, c } proper
λ_M = geocentric longitude of the Moon. b_1, c_1 coefficients

Note. Nutation:

1. Immutable sweeping the surface of a cone around P.
2. Semiellipse because of change in the Moon's orbit.
3. Displacements around the mean position within a distance $\Delta\omega$ because of the Sun and Moon.

3. General Remarks on the Methods of Observation and Reduction /17

3.1. Results of the Study of the Phenomenon

As we have already mentioned, the International Latitude Service was established in 1898, and renamed the International Polar Motion Service in 1963. This service collects the results of determinations of the astronomic latitude which are obtained in the international latitude observatories. Based on these data, this service computes and publishes, within a very short period of time, temporary values of the x,y coordinates of the pole, in its monthly periodical "Monthly Notes of the International Polar Motion Service."

Detailed description of the methods used for the computation of the temporary x,y coordinate values of the pole, and also improved values of these coordinates are published in its annual periodical "Annual Report of the International Polar Motion Service," approximately 2 years after the corresponding year.

The final elaboration of all the related data is done later, and the final values of the coordinates of the pole, x,y, are published.

Both the final and temporary values of the coordinates of the pole which are computed by the International Polar Motion Service are published with much delay, and therefore it is impossible for them to be used for the present needs of astronomy. In order to avoid this difficulty, the "Service Internationale Rapide des Latitudes" was founded in 1955; it computes and publishes the temporary values of the x,y coordinates of the pole /18 for the next month. For this purpose, the service, which is located in Paris, uses the results of the determinations of the astronomic latitude and of other astronomical observations related to time as well, which are carried out in 68 observatories distributed over the whole surface of the Earth.

In 1965, the "Service Internationale Rapide des Latitudes" merged with the International Time Office (Bureau International de l'Heure) which is also located in Paris.

This office publishes the monthly journal "Bureau International de l'Heure, Circuloire B/C," in which, among other things, the temporary values of x,y are contained, and also the predicted values of the x,y coordinates of the pole. Specifically, in the issue of this journal which circulates at the beginning of the month m + 2, the following are contained, among other things:

1. The temporary values of the x,y coordinates of the pole for the month m, which are computed on the basis of observations made during that month.

2. The predicted values of the x,y coordinates of the pole for the month m + 3, which are computed by means of extrapolation of the known polar orbit until the month m + 1.

In other words, we see that the temporary values of the x,y coordinates of the pole that are given by the International Time Office are published with a delay of only a month, and the predicted values refer to the month following that of publication.

The temporary values of the x,y coordinates of the pole that are computed by the International Time Office are published /19 in another journal also, edited from the same office, the "Bureau International de l'Heure, Circuloire D." Finally, a detailed description of the methods used for the computation of the temporary values of the x,y coordinates of the pole and also improved values of these coordinates are published 6 months after the end of the respective year in the annual journal "Bureau International de l'Heures, Annual Report," edited by the International Time Office.

3.2. Definition of Frames of Reference

In order to study the phenomenon of polar motion, we will have to define a frame of reference with respect to which we will obtain the instantaneous coordinates x,y of the instantaneous pole at each moment. In Chapter 2.1 we saw that in order to study the motion of the instantaneous pole we define a certain position of it as the basis of a frame of reference, and we compare the consecutive positions of the instantaneous pole with respect to the defined basis. In other words, an arbitrary position is sufficient to be defined as a basis. Of course, among the infinite positions we will choose the one that serves our purpose best.

For the choice of this position and the corresponding frame of reference, we observe from the first that because of the fact that the dimensions of the orbit of the pole on the surface of the Earth are very small (in an angle 1" with the center of the Earth's mass as a vertex, an arc of about 31 m length corresponds on the surface of the Earth), we can assume in a first approximation that the polar motion takes place on a plane tangent to the surface of the Earth at a point Δ properly chosen. If we also choose on this level a proper Cartesian coordinate system Δxy having an origin in the point Δ, then in order to define the position of the instantaneous pole Γ at each moment, it is enough to give the Cartesian coordinates x,y of the point Γ with respect to the system Δxy. /20

The best solution, of course, would be to choose as point Δ the point where the principal axis of inertia of the Earth intersects the surface of our planet. However, the position of the principal axis of inertia of the Earth within its mass is not exactly known. Moreover, this axis is continuously changing position within the mass of the Earth because of continuous distortions of the masses in the interior or on the surface of the Earth, which have as a consequence corresponding distortions of the inertial moments of the Earth. Because of this, we must find another way of choosing the point Δ.

The way to choose the point Δ has repeatedly changed in recent years. Thus, a frame of reference called Wauach 1900-1905 was used up to 1959.

Then point Δ was chosen to be the mean position of the pole during the time period 1900-1905, and the frame of reference defined this way was called the New System 1900-1905. At the same time, however, certain astronomical services were using a different frame of reference, having as the origin the so-called mean pole of epoch.

After the decisions of the recent general conventions of the International Astronomical Union (Prague, 1967) and of the International Union of Geodesy and Geophysics (Switzerland, 1967), a new way of defining Δ is now in use since January 4, 1968, and the thus-defined point Δ is known under the name Conventional International Origin, abbreviated C.I.O. or O.C.I.

The conventional international origin is defined in such a way that the corresponding values of the astronomical latitudes of the five international observatories of latitude operating today are equal to the values of the astronomical latitudes of these observatories that were used by the International Polar Motion Service in the year 1967. /21

Notice that the position of the conventional international origin on the surface of the Earth coincides with the origin of the new international frame 1900-1905. This is because the origin of the new international system 1900-1905 is defined to be the mean position of the pole during a certain time period (1900-1905), while the conventional international principle is not related to the position of the pole during a certain time period, but is defined by means of the values of the astronomical latitudes of the International Latitude Observatories. Therefore, if we need to reconsider in the future, for any reason, the values of the coordinates of the pole during the years 1900-1905, then the position of the origin of the new frame of reference will change, while the conventional international principle will remain unchanged,and therefore it will coincide with the origin of the new frame of reference 1900-1905.

The directions of the x,y axes are chosen in such a way that the x-axis is tangent to the meridian of Greenwich and its positive course is pointing toward Greenwich, and the y-axis is perpendicular to the x-axis and its positive course points toward the corresponding meridian at the value of longitude $\lambda = 90°$.

With respect to this system, the temporary values of the x,y coordinates of the pole are computed and published, starting from January 1, 1965, by the two international services, that is the International Polar Motion Service and the International /22
Office of Time.

However, for the computation of the temporary values which are published by the International Polar Motion Service, only the observations of latitude which are made in the five international latitude observatories located on the 39°8' parallel are used, while for the computation of the temporary values that are published by the International Office of Time, both the observations of latitude and those of time which are made in the 68 observatories that cooperate with this office are used. Therefore, the temporary values of x,y that are published by these two services do not, in general, agree. In any case, the differences occur partially because of the difference in methods used by the two services for the correction of the systematic errors of the observations and calculation of the x,y values.

Notice that the results of the observations made in the above-mentioned 68 observatories cooperating with the International Office of Time are also used by the International Service of Polar Motion for the computation of the final values of the x,y coordinates of the pole. Obviously, because of the variety of instruments and methods of observation used in these 68 observatories, this material is much less homogeneous than the total given by the observations made in the international latitude observatories.

3.3. General Method of Observation

The determination of wobble (general disturbance) includes precise measurements of the latitude according to the figure in Paragraph 2.1. The meridian civile gives the fundamental method for the determination of latitude. The instrument we use is a telescope which can rotate around a horizontal axis and which is oriented from East to West. /23

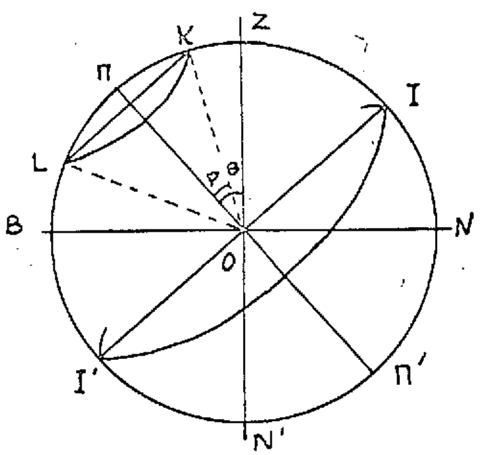


Fig. 1.

Let Z_η be the value of zenith distance at the upper culmination (uc) and Z_l be the value of zenith distance at the lower culmination (lc) 12 sidereal hours later. We shall obtain:

$$\theta = \frac{1}{2}(Z_L + Z_u), \quad p = \frac{1}{2}(Z_L - Z_\eta).$$

where $\theta = 90 - \phi$, $p = 90 - \delta$ (co-declination).

The above method is known as the fundamental method of determination of latitude and declination because the value of one of these is obtained independently of the other. This method has the disadvantages of the upper and lower aiming of the same star. In other words, we must work with stars which are always visible, and these must be such that they have early upper culmination, so that we will be able to observe them at the lower culmination in the early morning hours. This can be more easily done during the winter, when the nights are longer. However, this is one more disadvantage for the observations. This method consists of calculating the difference /24 between the almost equal zenith distances of two stars that pass from the meridian a few minutes apart, one of them north and the other south of the zenith, and not far from it.

The angle θ is connected with the two crossings with the meridian, having the relations $\theta = p' + Z'$, $\theta = p - Z$ (Fig. 2).

From these, we conclude that

$$\boxed{\theta = \frac{1}{2}(p' + p) + \frac{1}{2}(Z' - Z)}$$

The value of the difference $(Z' - Z)$ can be measured, and the declination of the pair of stars can be known. Talcott's method is not a fundamental one. However, it is the most commonly used because of the following advantages:

1. The use of zenith stars reduces the optical diffraction error. This follows from the formula of diffraction.

2. The computation of the small angle $(Z - Z')$ by the use of a micrometer is much more precise than the calculation of Z_U, Z_L from a graduated circle.

The zenith telescope is the classical instrument in Talcott's method. By aiming from an exact horizontal plane (disc), the telescope is set in the proper zenith distance for the star which first approaches the meridian. When this star passes the meridian crosshair, its distance from the center of the cross is measured by a micrometer (great advantage). Then, the telescope rotates 180° around a vertical axis so that it points north of the zenith. Of course, if it were pointing north before, and if it is necessary, the telescope is set again until the plane (horizontal disc) becomes horizontal. The important thing is that the angle between the horizontal disc and the telescope must remain constant. Then the telescope /25 (i.e. as long as the angle remains constant) is set at the same zenith distance as before, but on the opposite side of the zenith. The micrometer measurement is repeated for the second star, and comparison of the two measurements gives the value of $(Z' - Z)$ without the need for dependence on a graduated circle (the vertical disc), but only on a smaller displacement. In other words, the main advantage of Talcott's method is that we do not use the graded disc but only the micrometer knob. This way, we avoid arbitrary errors of division and reading. Obviously we will have errors of the micrometer knob, which will be mentioned later in more detail.

Another important advantage of Talcott's method is the elimination of diffraction because of the value obtained.

Moreover, this method makes many more stars available for observations, and finally, it does not require a star that culminates very near the zenith for equal accuracy. Some alterations have been made on the zenith telescopes. The horizontal level in the photographic zenith tube (PZT) (cylinder) has been replaced by a free mercury surface by which the image of the star is reflected on a photographic metal-coated plane which rotates around a vertical axis. We take several exposures of the single star before and after rotation when the distance 2Z between the images is computed by a "measurement machine."

The complementary latitude θ (co-latitude = 90 - ϕ) equals p ± Z where the zenith distance has the (+) sign if the star is north of the meridian, and the (-) sign when it is south of the meridian.

The impersonal astrolabe is based on a completely different method from that of Talcott. A ray coming from a star and a ray /26 reflected on a mercury surface enter the lens of the telescope through a 60° prism, making two images. When the two images coincide, the zenith distance of the stars is 30° (Danjon, 1958). Determination of time and latitude are dependent on each other and the knowledge of the right ascension of a star is required to be added to the declination (methods of latitude determination). The above does not hold for the photographic zenith tube (PZT).

3.4. General Methods of Reduction

The arithmetic value of the disturbance (wobble) is on the order of 0".1. The probable error of observation of a star is on the same order. Therefore, for studying the wobble, we must limit the probable error of observation. About a thousand observations were done monthly in a station, on the average, so that the probable error was determined to be about 0".01, provided the errors were just random. A comparison between the latitude observed in Washington and the results of the I.L.S. stations showed a difference on the order of 0".1 (Fig. 7.4). These and other proofs showed that there are significant systematic errors.

Reduction from the I.L.S. stations follows a certain process. The main difficulty is due to the pace of the micrometer knob and the declination of the star under observation. Nothing is known a priori with satisfying precision for reduction, and in the final analysis, the latitude observations give information for the various corrections. The corrections of the pace /27 of the knob include an annual term which can be as wide as the annual motion of the pole. Melchior (1957) thought that the accuracy of the observations could be known in geophysics from the accuracy of the pace of the knob.

In order to make the errors due to the pace of the knob smaller, the group of latitude stars (which are arranged in Talcott's pairs) is chosen in such a way that the sum of all the small measurements in a certain night is the smallest possible. Unfortunately, the coordinates of the latitude stars are so influenced by precession after a decade, and some of the stars from the Talcott pairs are so unsatisfactory, that they must be recomputed. Changes in the catalogues were made in 1912, 1922.7, 1935 and 1955. In Cohn's catalogue, which was used from 1899 to 1935, some declinations were known to have an error.

The most accurate catalogue is that of Boss; it has been used since 1935, but it also includes some errors. Finally, the number of stations has changed from three (1922.7-1935) to six (1901.7-1906).

All these changes have obviously caused some inhomogeneity in the observations.

The final coordinates which are published with a few years' delay after the temporary coordinates include corrections for the pace of the knob and the declination of the stars and, according to Melchior, the temporary coordinates are more trustworthy than the ones obtained after correction.

According to the I.L.S. notes, the estimated reduction to $(90 - \phi)$, i.e. $\Delta\theta_u$ from a station u west on a longitude λu is given by the relation: /28

$$-\Delta\vartheta_u(t) = x(t)\cos\lambda_u + y(t)\sin\lambda_u + Z(t)$$

where $x(\equiv m_1)$ is the displacement of the pole of rotation toward Greenwich, $y(\equiv m_2)$ is the displacement along a direction 90° west of Greenwich and Z is the correcting term which was introduced by Kimura. The coordinates x,y and Kimura's term Z are determined by the method of least squares on the $\Delta\theta_u$ (difference) of the latitude observations in all stations.

The physical meaning of Kimura's term, of which the absolute value is not more than a few tenths of a second of an arc, remains unknown. Kimura's term has a polar change in latitude, i.e. as if the latitude were increasing or decreasing the same way in all I.L.S. stations. Usually this change is on the order of 0".03. This term eliminates errors (1) in the values of the motion and declination of the stars under observation, (2) coming from the fact that we do not take into consideration differences in these stars, and (3) of the fundamental constants of astronomy (nutation, annual reduction, oblique course, etc.) The term does not disappear if we substitute the PZT for the zenith telescope.

It has been observed that the term Z(t) tends to have the same point in both hemispheres. (Kimura's latitude reductions have the opposite sign for the northern and southern hemisphere.) This is the expected difference from a displacement in the center of mass of the Earth, for example, for a seasonal flow matter vertically to the equator. This problem requires further study.

The rest of the changes that are not correlated among the stations are called "local Kimura terms" of certain direction. /29

The latitude computation is relatively easy, except the correction above, and is given by the formula

$$-\Delta\varphi_u(t) = x(t)\cos\lambda_n + y(t)\cos\lambda_n + Z(t).$$

The local change is correlated with wind, pressure, and other meteorological changes. The optical diffraction is minimized by the use of zenith stars, but this zenith diffraction can give wrong results. It is useful at this point to distinguish the "room diffraction" which is connected with the conditions of the immediate environment around the telescope, and with the whole atmospheric diffraction. Pzzbyllok (1927) compared the latitude measured in the Washington Naval Observatory with that determined by I.L.S. After that, he computed the reduction and compared it with the local wind direction, and he found that during a northerly wind, the latitude of Washington increased by 0".02, while during a southerly wind, it decreased by 0".02.

He attributed this difference to the "room diffraction."

Pzzbyllok suggested that the monthly values at the stations could have an error on the order of 0".25, while the annual values could have an error of about 0".1 (!)

During the last 2 years of research at the original Tschardyui station, the average latitude was found to have a change on the order of 0".1. Lambert (1922) attributed this abnormality in the observations to the diffraction caused by change in atmospheric conditions as, for example, the displacement of the Amu Darya River from its former position about 3 km /30
toward the station.

It had been expected that under normal conditions the seasonal diffraction change would have been eliminated by Kimura's term, and that the relation between room diffraction and the term Z would have been found. There is, however, no guarantee that a significant error due to diffraction does not remain in the computed annual latitude change. Also, a tendency to change can exist in the mean term of the zenith distance accompanied by a change in climate during the 20th century. However, this requires further study (as does sideways diffraction).

Of course, the use of some small astronomical results is permitted for the determination of latitude. These include Battermann's result for the oblique course and small nutational terms due to the disturbance by perturbation of the Earth's orbit caused by Saturn and Jupiter. The declination is also influenced by the wobble of the rotational axis with respect to the fixed axis (the fixed axis in space if no external torque exists).

Every geophysical phenomenon causing wobble should be connected not only with latitude change, but also with declination change. In the computation of latitude, according to Talcott's formula, the declination is not corrected for wobble (sway) but wobble (sway) is dependent on the change of latitude with time, and the precession constant H (= 0.003) is insignificant. Chandler did experimental research on sway, but he could not detect it. "The comparison of the absolute and the different definition shows that the phenomenon refers completely to zenith change and is not separated in a change of the zenith and of the astronomical pole (at the same time)." /31

Concluding the chapter on generalities, we can observe the following about the needs in observations:

1. The Kimura term has not been investigated fully enough, so we do not know what it depends on and how it acts. From our hypotheses, the seasonal change of diffraction should be eliminated by the Kimura term, and the relation of this term to the "room diffraction" should have been found. This has not been achieved, and requires further study.

2. Besides the Kimura term, the concept of "local Kimura terms" was also introduced, terms that are generally dependent on local meteorological changes. We have shown that these changes have sufficiently large values. Therefore, the results from the stations to be compared should be reduced to similar conditions. Besides this, and the examination of "room diffraction," the problem also depends on general meteorological change in a place, due to changes in conditions, e.g., a displacement of a river or a change in climate with succeeding centuries. This also needs further study.

3. Within the above framework, the general problem of "optical diffraction" appears. In other words, we deal with the general problem of diffraction, which is of interest, examined either as general atmospheric diffraction or as "room diffraction," and there is much research to be done on this, because of the phenomena of magnetohydrodynamics in the last /32
layers of the atmosphere, which are believed to affect optical radiation. Finally, the problem of sideways optical diffraction always exists.

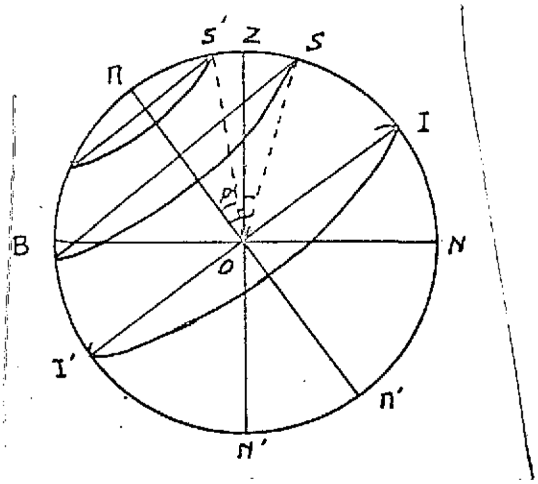


Fig. 2.

4. We have already mentioned that some displacements in the center of mass of the Earth are believed to be of physical causes, from the term Z(t). In other words, the relation between the core and crust requires deeper study and examination and is of decisive importance for the whole problem of polar motion. So we see that there are still some unsolved problems. The positive research for answers gives a measure of the approximation of the values found.

5. We should also regard with some skepticism the homogeneity of the static analysis of all stations, since each one has some geophysical errors. For example, take the phenomenon of earthquakes that is more common for the area of Mizusawa and less common for Ukiah and Carloforte. Also, other, more general phenomena such as the change in the flow of a river, vegetation, large technical construction, etc. must be taken into account.

4. DETAILED DEVELOPMENT OF THE METHODS OF CORRECTION AND REDUCTION

4.1. Introduction

In this chapter we shall study in detail all the methods that are used for the observation of stars and the reduction of the results. Together with development of the methods, examples and tables will be given, so that this chapter will constitute the basis for further study. After these, the orbit of entering quantities of the pole will be given in a figure, with respect to the new system 1900-1905, which coincides with the CIO for the time being, and the motions of the pole will be given as a summary, which constitutes the final purpose of this work. /33

For the study of the methods, information is given for the final results of the first volume of "Annual Report." This volume includes the results of the latitude observations obtained during 1962 in the stations and observatories connected with the International Polar Motion Service, but it does not include the results of the time observatories which refer to polar motion. We describe below the results from 32 stations, i.e. five I.L.S. latitude observation stations, and 27 other stations with 32 instruments.

This study can be separated into the following chapters:

The first chapter, that includes descriptions of the original information given from the five I.L.S. stations and the methods of reduction used by the Central Bureau. This same chapter also includes the results of the latitude check for the five I.L.S. stations according to the information of the Scientific Council of IPMS. /34

Chapter 2 includes brief descriptions of the results from the latitude observations obtained in the independent stations and sent to Mizusawa, or the results published in the "bulletins" of the related stations, together with the differences between the observed latitude and the computed one.

The coordinates of the pole are obtained by using the available information from all sources. Also, they are studied so that they become more appropriate for the computation of the coordinates of the pole, in accordance with those already obtained by the I.L.S. In coordinating the results of the independent stations and those of the five I.L.S. stations, some problems seem to exist which must be solved, such as the determination of the mean latitude in the new system 1900-1905 of I.L.S., the use of Kimura's term Z, and the comparison of the star catalogues used by the respective stations. Because of this, the determination of the coordinates of the pole was defined in accordance with the new system 1900-1905 of the I.L.S. only by the results of the five I.L.S. stations.

4.2. Chapter 1: Results from the Five I.L.S. Stations Lying on the 39°8' North Parallel

4.2.1. Star Catalogue and Program of Observation

During 1962, the first IPMS year, the following five I.L.S. stations continued the same observations of latitude, with the usual zenith telescopes, from the previous year:

Mizusawa, Japan	$\lambda = -9^h$	24^m	31^s	
Kitab, USSR	$\lambda = -4^h$	27^m	21^s	/35
Carloforte, Italy	$\lambda = -0^h$	33^m	15^s	
Gaithersburg, Mar., USA	$\lambda = 5^h$	8^m	48^s	
Ukiah, USA	$\lambda = 8^h$	12^m	50^s	

No change has been made in the program of observation and in the stars taken in 1955. The catalogue includes 144 stars and forms a total of 72 pairs which are arranged in 12 groups, each of which includes 6 pairs and occupies 2 hours of the right ascension, as is shown in Table 1. The stars have been placed in order according to the general catalogue of Boss. In placement, the centennial changes are given, and one half of the centennial changes of the annual changes, i.e., the changes of the centennial changes. These will be denoted as CV (centennial changes) and SV (eternal changes). The same motions are given for 1 year. For the right ascension, 100 values of the annual change and 50 values of the eternal change are given in the general catalogue of Boss, and also CV and SV for 1950. The CV and SV for the declination are computed by the following formula:

$$(CV)_\delta = \frac{d\delta}{dT} = \eta_0 \cos \alpha_0 + 100 \mu'$$

$$(S.V.)_\delta = \frac{1}{2} \left(\frac{d\nu}{dT}\right)_0 \cos \alpha_0 - \frac{1}{2} \sin 1'' (m_0 \eta_0 \sin \alpha_0 + 200 \eta_0 \mu \cdot \sin \alpha_0 + 500 \mu^2 \sin 2\delta_0 + \eta_0^2 \sin^2 \alpha_0 + \alpha \eta \delta_0).$$

TABLE 1. MODEL OF STAR CATALOGUE OF I.L.S.

Group	Pair	G.C No	Apparent Magnitude	Spectral Formula	Right ascension 1950.0	Annual Change	Eternal Change	3rd Term	μ	Declination 1950.0	Annual Change	Eternal Change	3rd Term	μ'
VII	37	15273	6.59	K5	$11^h 04^m 05^s.113$	$317^s.98$	$-0^s.565$	$0^s.010$	$0^s.0016$	18° 00′ 29″.26	−1948″.603	−5″.185	0″.02	0″.004
		15492	6.66	A3	11 13 22.609	353.96	−3.375	.063	−.0053	60° 12′ 56″.71	−1964.828	−4.785	.02	.057
	38	15824	6.56	K0	11 29 32.659	320.45	−1.185	.018	+.0001	36° 31′ 25″.90	1989.988	−2.670	.02	.041
		16088	6.81	G5	11 40 58.701	317.48	−1.390	.022	+.0018	42° 00′ 00″.05	1998.759	−1.495	.01	−.069
	39	16199	5.76	F5	11 47 06.089	311.69	−1.025	.017	−.0094	35° 12′ 34″.27	−2001.285	−0.850	.01	−.021
		16445	5.07	A3	11 59 34.806	304.61	−1.290	.023	−.0295	43° 19′ 22″.63	−1997.555	0.385	.01	−.116
	40	16767	6.00	K2	12 15 01.980	295.89	−1.725	.034	+.0038	53° 28′ 09″.86	−2005.348	1.840	.00	.102
		16866	6.02	A0	12 19 40.108	301.51	−0.520	.018	−.0046	25° 03′ 03″.70	−1998.282	2.305	.00	.010
	41	16958	7.04	A0	12 24 09.452	287.07	−1.705	.036	+.0015	55° 26′ 09″.65	−1992.034	2.625	.00	−.252
		17224	6.47	K0	12 36 35.187	297.98	−0.390	.012	−.0036	22° 56′ 03″.41	−1970.744	3.365	.00	.049
	42	17329	6.34	F5	12 42 05.738	283.18	−0.995	.020	−.0026	44° 22′ 34″.57	−1951.020	4.185	−.02	−.338
		17517	6.26	A2	12 51 49.893	286.50	−0.605	.014	−.0077	33° 48′ 17″.37	−1931.017	5.085	−.02	−.069
VIII	43	17767	6.04	F5	13 03 55.003	292.34	−0.240	.010	−.0055	21° 25′ 16″.92	−1910.612	6.265	−.02	−.062
		17919	6.74	G0	13 11 34.420	245.53	−1.010	.024	+.0135	36° 28′ 22″.17	−1845.240	5.935	−.04	−.009
	44													

/36

where α_0 and δ_0 are the right ascension and the declination for 1950.0 which are given in the General Catalogue, and m_0, n_0, $(dn/dT)_0$ are the centennial values for 1950 which are computed by the following formulas, given earlier by Newcomb.

$$
\begin{aligned}
m &= 4608''.506 + 2''.7945\,T + 0''.00012\,T^2 \\
n &= 2004''.685 - 0''.8533\,T - 0''.00037\,T^2 \\
\frac{dy}{dT} &= -0''.8533 - 0''.00074\,T
\end{aligned}
$$ /37

where T is measured in tropical years from 1900.0. The arithmetic values of m, n, dy/dT for 1950 will be:

$$
\begin{aligned}
m_0 &= 4609''.903 \\
n_0 &= 2004''.258 \\
\frac{dy}{dT} &= -0''.8533
\end{aligned}
$$

Applying the above values of the changes of CV, CS for the catalogue, we will have the fact that the mean positions of the stars for any other epoch t are computed by means of the following formula:

$$
\begin{aligned}
\alpha_t &= \alpha_0 + (C.V.)_\alpha\, t + (S.V.)_\alpha\, t^2 + (3^{\text{rd}}\ \text{term})_\alpha\, t^3 \\
\delta_t &= \delta_0 + (C.V.)_\delta\, t + (S.V.)_\delta\, t^2 + (3^{\text{rd}}\ \text{term})_\delta\, t^3
\end{aligned}
$$

where t is measured in tropical centuries from 1950.

The duration of the days of observation for every composition, consisting of three consecutive groups which are symmetrical with respect to midnight, was a month. The groups were observed during the following periods:

Jan 6 - Feb 5	IV - V - VI	July 7 - Aug 5	X - XI - XII
Feb 6 - Mar 6	V - VI - VII	Aug 6 - Sep 5	XI - XII - I
Mar 7 - Apr 6	VI - VII - VIII	Sep 6 - Oct 5	XII - I - II
Apr 7 - May 6	VII - VIII - IX	Oct 6 - Nov 5	I - II - III
May 7 - June 6	VIII - IX - X	Nov 6 - Dec 5	II - III - IV
June 7 - July 6	IX - X - XI	Dec 6 - Jan 5	III - IV - V

4.2.2. Static Analysis of the Observations /38

14,679 pairs of stars were observed at the five I.L.S. stations during the period from January 6, 1962, to January 5, 1963. The monthly numbers together with the values of the nights of observation are given in Table 2.

4.2.3. Correction of the Telescope Errors

The telescopes were well arranged and no correction was referred during 1962. Certain errors of observation are shown in Table 3. The values of the inclination of the horizontal axis i_c and i_w are the mean monthly values from every night in 1 month in the unit of /39
time (sec), except for Carloforte, for which the values are given in the unit of division of the horizontal disc. In Gaithersburg, one special arrangement was invented for the elimination of the sideways bending of the telescope by raising the telescope from the horizontal axis by a quantity equal to the bending. Therefore, the values of inclination for the same number are more regular and more comparable to the ones of other stations. This method was proved by E.L. Williams in 1932, and it has been used from that time on in Gaithersburg.

EXAMPLE: TABLE 2.

Month	Mizusawa		Kitab		Carloforte Gaithersburg Ukiah	
1962	Night	Pair	Night	Pair	Night	pair
Jan.	13	102	2	19		
Feb.	14	107	9	115		
Mar.	16	148	6	107		
Apr.	19	216	5	89		
May	17	176	9	147		
Jun.	13	182	20	315		
Jul	14	126	20	326		
Aug.	15	108	23	385		
Sept	14	119	16	287		
Oct.	19	157	10	134		
Nov.	14	134	9	114		
Dec.	13	162	12	163		
Total	181	1737	141	2201		

EXAMPLE: TABLE 3.

Mizusawa 1962

Month	Inclination i_e	i_w	Date of Observation	Bending b	Aiming c	Azimuth x_e	x_w
Jan	$0^s{,}06$	$-0^s{,}01$	9	$2^s{,}38$	$2^s{,}52$	$0^s{,}04$	$-0^s{,}24$
			25	2,44	2,56	-0,39	0,15
	0,03	-0,06	8	2,22	2,41	0,17	-0,20
			21	2,73	2,86	0,10	0,07

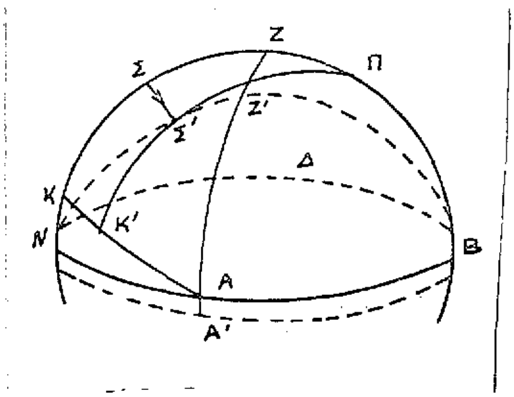
Gaithersburg 1962

Month	Inclination			
	Before the Observation		After the Observation	
	i_e	i_w	i_e	i_w
Jan	$-1^s{,}03$	$1^s{,}03$	$-1^s{,}01$	$1^s{,}01$
Feb.	-1,00	1,00	-1,01	1,01

A further explanation of the errors of inclination in aiming (direction) and of azimuth is required in order to understand the corrections of these errors. /40

4.2.3.1. Errors of Inclination

An error of inclination is caused because of the nonhorizontal axis of rotation of the telescope tube. The inclination i of the horizontal axis of the telescope tube with respect to the horizon has an influence on the exact time of observation of the culmination of a star. We shall try to explain this influence.



Suppose that we have the celestial sphere, the equator and the points of horizon A, B, Δ, N. The horizontal axis does not lie on the horizontal plane. If we aim with this axis at the point A on the east, the axis actually would pass from A' ≠ A so the inclination of the horizontal axis will be (AA') = i = (ZZ'). Therefore, the optical axis of the telescope tube rotating around the horizontal axis, instead of describing the meridian, will describe a great circle through the points B', Z', N having as a pole the point A' (point Z' is east of point Z).

Therefore, if we aim at a star, we will take as time of culmination the time at which the star is passing from the meridian of the instrument. The meridian of the instrument is different from the meridian of the place of observation, and the star will culminate in less time than the time it needs to describe the arc (ΣΣ'). We draw the hour circle from point Σ'. This circle intersects the equator at a point K'. We will have:

$$(\Sigma\Sigma') = (KK')\cos\delta \qquad \text{but} \qquad (\Sigma\Sigma') = (ZZ')\cos Z\Sigma \qquad (1)$$ /41

$$\text{So } (KK')\cos\delta = (ZZ')\cos Z\Sigma = i\cos Z\Sigma \qquad (2)$$

Relations (1) follow from the ratios.

It is also true that

$$(\Pi B) = \varphi = (KZ) \quad \text{and} \quad (Z\Sigma) = (KZ) - (K\Sigma) = \varphi - \delta \Rightarrow$$

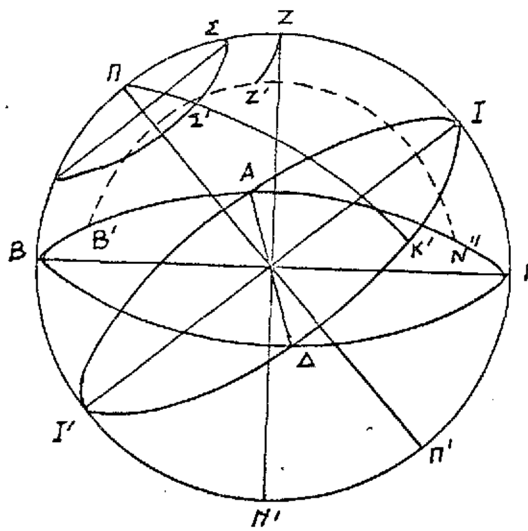
$$(KK')\cos\delta = i\cos(\phi - \delta)$$

So the arc (KK') expressed in time is given by the formula $(KK') = i\cos(\phi - \delta)\sec\delta$.

This formula gives the correction from the error of inclination i which we have to apply in order to bring the observation from the meridian of the instrument to the meridian of the place of observation. Exactly this error is given by components on the previous table, so that we will correct our results, eliminating errors caused from the nonhorizontal position of the horizontal axis.

4.2.3.2. Errors of Aiming (Direction)

This error is defined as the difference of the angles C between the angle of the optical axis of the telescope tube and its horizontal axis and 90°. That is, this error is due to the optical axis not being perpendicular to the horizontal axis. Because of that, the optical axis will, during the rotation of the telescope tube, describe a small circle N'Z'B' parallel to the meridian of the place of observation and lying farthest east of it.



Suppose we have the celestial sphere, its equator II' and the horizon. We will have:

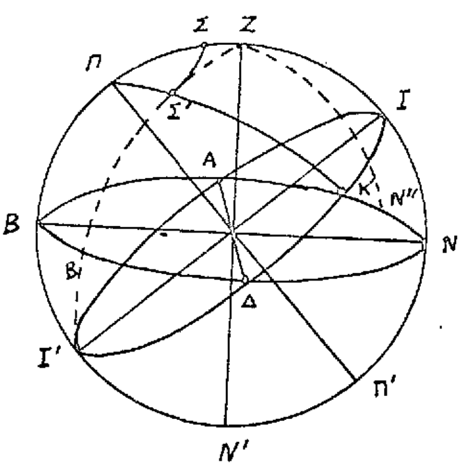
$$(NN'') = (BB') = (ZZ') = C$$

Therefore, when we aim at a star and see it in culmination, the star actually is culminating on the meridian of the instrument and not on the meridian of the place. Because of that, a correction is needed to reduce the meridian of the instrument to the meridian of the place of observation. If Σ, Σ' are the points of culmination of the star on the meridian of the place and the meridian of the instrument, respectively, then the time which is necessary for the star to describe the arc (ΣΣ') gives the time before which the culmination will occur. In order to find the necessary time for the star to describe the arc ΣΣ', we must find the hour angle of Σ'. /42

For this purpose, we write the hour circle of Σ' which intersects the equator at the point K'. Then we will have C = = (ΣΣ') = (IK') cos δ. Therefore, the arc (IK') = C sec δ, and it gives the correction which we have to apply in order to find the moment of crossing the meridian of the place.

4.2.3.3. Errors of the Azimuth

The error of the azimuth is defined as the angle θ between the optical axis and the meridian which can be formed because of noncoincidence. If we take the east end of the telescope tube, this will result in that, instead of intersecting the horizon at the point A, it will intersect it at another point A'. It will be $(AA') = \theta$. During its revolution, the optical axis, instead of describing the meridian of the place of observation, will describe another great circle B'ZN'. Therefore, as the moment of culmination we will not consider the moment at which the star crosses the meridian of the place, but the moment at which it crosses the meridian of the instrument. The arc $\Sigma\Sigma'$ gives the time interval during which the culmination will be observed earlier. (It must be noted that the describing of a great circle is a difference in order to avoid confusion with the error of aiming.) Here the axis is perpendicular, but it forms an angle.) /43



As with the previous errors, in order to determine this time we consider the hour circle of point Σ'. We will have:

$$(\Sigma\Sigma') = (KK') \cos \delta$$

$$\frac{(\Sigma\Sigma')}{KK'} = \cos \delta$$

however, $(\Sigma\Sigma') = (NN'') \cos N\Sigma = \theta \sin (\phi - \delta)$.

Therefore, $(KK') = \delta \sin (\phi - \delta) \sec \delta$, where the arc (KK') is expressed in time and λ is given with component (θ east, θ west).

If we assume that the above errors take place, which is true, the total correction which we must apply will be (Mayer formula):

$$\Sigma = C \sec \delta + i \cos (\phi - \delta) \sec \delta + 2 \sin (\phi - \delta) \sec \delta$$

If we develop the $\sin(\phi-\delta)$, $\cos (\phi-\delta)$ and put

$$\left.\begin{array}{l} K = i\cos\phi + \lambda\sin\phi \\ \lambda = i\sin\phi - \lambda\cos\phi \end{array}\right\} \Sigma = K + \lambda\tan\delta + C\sec\delta \quad \text{(Bessel)}$$

From the above formulas, consequently, we have the fact that in order to avoid the errors of the instrument, we prefer to aim at stars which have a small value of declination.

In the above manner we correct the observations, if we take into consideration the error due to sideways bending of the instrument.

4.2.4. Atmospheric Terms

The atmospheric terms are usually measured every hour at the time intervals of the latitude observations. These are summarized in Table 4. Abbreviating all the mean monthly values for every group, that is, nightly values, morning, and intermediate, are arranged in the table. The atomic hour values of Tex, Ttel and B were used for the determination of the diffraction and the correction of temperature for the constants of the instrument in the conversion series. We distinguish the following quantities: /44

nTex, sTex	External temperature in centigrade with respect to the north and south side.
Tex	Mean value of the above two quantities
ΔTex	Hourly change of Tex.
nTex - sTex	Change between external temperatures, north minus south.
nTin, sTin	Internal temperature with respect to north and south side.
Tin	Mean value of the above quantities
nTin - sTin	Difference between internal temperatures, north and south
Ttel	Telescope temperature
Tex - Tin	Difference between external and internal temperature
Tex - Ttel	Difference between external temperature and telescope temperature.
Tin - Ttel	Difference between internal temperature and telescope temperature.

B	Barometric reading in mmHg corrected for both the temperature and gravity at Mizusawa and Ukiah, but only for temperature at Kitab, Karloforte and Gaithersburg.
AB	Hourly changes of B
SE	Disturbance of the scale images (1-4) (better-worse)
ST	Stability of the scale images 1-4 (better-worse)
W.V.	Velocity of the wind near the telescope in m/sec
W.D.	Direction of the wind estimated on the right-hand side from North
NS - comp	Mean value of the component of the wind velocity from N → S during the observations. /45
EW - comp	Mean value of the component of the wind velocity from E → W during the observations.
Humidity	Relative humidity during the observations

The following example has a sample of measurements and corrections of the above magnitudes at Mizusawa in January, 1962.

MIZUSAWA, JANUARY 1962.

Group	ηTex			sTex			Tex			ΔTex			ηTex-sTex		
IV, V, VI	e	i	m	e	i	m	e	i	m	e	i	m	e	i	m
	-2°62	-3°79	-3°74	2°74	-3°91	-3°91	2°68			0°37			0°12		

ηTin			sTin			Tin			ΔTin			ηTin - sTin.		
e	i	m	e	i	m	e	i	m	e	i	m	e	i	m.
-2°56	3°75	-3°76	-2°54			-2°55			-0°40			-0°02.		

T_{tel}	ΔT_{tel}	$T_{ex} - T_{in}$	$T_{ex} - T_{tel}$	$T_{in} - T_{tel}$
B (mm)	ΔB (mm)	SE	ST	WD
WV (m/sec)	NS - comb (m/sec		EW - comb (m/sec)	

Summarizing what has been said up to now, we note the following:

1. In order to eliminate the errors, working and using Talcott's method we observed 72 pairs (144 stars) which we separated into 12 groups (each group has six pairs). Each month we observe three groups, so that by finding the mean values of a group, we eliminate random errors.

2. We correct our observations for the instrumental errors. /46
These corrections were defined, justified and applied so that our results will be corrected and ready for reduction.

3. The various atmospheric terms were described, as well as the magnitudes which they produce; we take these into consideration during reduction. Therefore, we can now develop the methods of reduction of the observations.

4.2.5. Methods of Reduction

The individual values of the latitude observations were computed by the formula:

$$\varphi = \frac{1}{2}(\delta_s + \delta_n) + \frac{1}{2}(Z_s - Z_n)$$

where δ, Z are the apparent declination and the apparent zenith distance of the star. The indicators n and s show the north and south stars of the pair. The apparent declinations of the stars were computed up to 0".001 (thousandth of a second of the arc) by the following formula, where the correction for the small amount of second class terms was also taken into consideration:

$$\delta_\tau = \delta_t + \tau\mu' + Aa' + Bb' + Cc' + Dd' + A^2 \sin 1'' \cdot \frac{1}{2}\left(-\frac{m}{n}\sin\alpha_t - \sin^2\alpha_t \cdot \tan\delta_t\right) +$$
$$+ AB\sin 1''\left(\frac{m}{n}\cos\alpha_t - \frac{1}{2}\sin 2\alpha_t \cdot \tan\delta_t\right) + B^2\sin 1''\left(-\frac{1}{2}\cos^2\alpha_t \cdot \tan\delta_t\right) +$$

$$+AC\cdot\sin 1''\left(\frac{1}{2}\sin 2\alpha_t-\sec\delta_t-\frac{m}{n}\cos\alpha_t-\sin\delta_t-\tan\varepsilon_t\cdot\cos\alpha_t\cdot\sin\delta_t\right)+$$

$$+AD\sin 1''\left(\sin^2\alpha_t\cdot\sec\delta_t-\frac{m}{n}\sin\alpha_t\cdot\sin\delta_t+\cos\delta_t\right)+BD\sin 1''\left(-\frac{1}{2}\sin 2\alpha_t-\sec\delta_t\right)+$$

$$+C^2\sin 1''\left(-\frac{1}{2}\cos^2\alpha_t\cdot\tan\delta_t-\tan\varepsilon_t\cdot\sin\alpha_t\cdot\cos 2\delta_t-\frac{1}{2}\tan^2\varepsilon_t\cdot\sin 2\delta+\right.$$

$$\left.+\frac{1}{2}\sin^2\alpha_t\cdot\sin 2\delta_t\right)+$$

$$+CD\sin 1''\left(\frac{1}{2}\sin 2\alpha_t\cdot\tan\delta_t-\frac{1}{2}\sin 2\alpha_t\cdot\sin 2\delta_t+\tan\varepsilon_t\cdot\cos\alpha_t+\cos\delta_t\right)+$$

$$+D^2\sin 1''\cdot\frac{1}{2}\left(\cos^2\alpha_t\cdot\sin 2\delta_t-\sin^2\alpha_t\cdot\tan\delta_t\right)$$

where α_t, δ_t are the mean positions at the closest beginning of the Bessel year t, which is measured in tropical centuries from 1950.0. τ is the epoch of observation, measured in tropical years from the closest beginning of the Bessel year t, and not exceeding half the year. μ' is the proper motion in declination given in the star catalogue. a', b', c', d' are the constants of the star in right ascension at the closest beginning of the Bessel year t. /47

ε_t is the mean obliquity of the ecliptic determined by the formula:

$$\varepsilon_t = 23°26'44,''84 - 46,''850\,t - 0,''0032\,t^2 + 0,''00181\,t^3.$$

and m, n are computed by the relations already given in 4.2.1.

It is noted that we use the linear form of transformation formulas appropriately reformed, because we are computing the apparent positions of the same stars at specific instants, so the values of a', b', c', d' will remain constant for all cases, and only the computation of the respective Bessel numbers A, B, C, D is necessary. Among these, C and D were computed by the following formula, instead of the values which are given by the astronomic ephemeris.

$$C = 1189,''80\,(-\dot{Y}_0 - \dot{y} + 0{,}0000553) + 0,''0092\,t$$

$$D = -1189,''80\,(-\dot{X}_0 - \dot{x} + 0{,}0002815) + 0,''0028\,t$$

where $\dot{x}_0$, $\dot{y}_0$ are the components of the velocity of the Sun referring to the true equator and to the equator of the date. $\dot{x}$, $\dot{y}$ are the components of the center of gravity of the solar system.

It still remains to examine the reduction of the differences in zenith distances, in order to apply the general formula. The difference in zenith distances of a pair of stars is computed by the formula:

$$\frac{1}{2}(Z_s - Z_n) = \frac{1}{2}(M + B\,T_{tel})\left[R_E + (IS)_E + (WI)_E - R_W - (IS)_W - (WI)_W\right] +$$

+ 1/2 (plane of correction) +

+ 1/2 (spherical correction) /48

+ 1/2 (differential refraction)

where M is the micrometric constant, that is, the value of one revolution of the micrometer in seconds of arc. This has to be known in order to be able to measure the revolutions of the past revolution in seconds of arc.

β is the temperature coefficient of the micrometric constant.

Ttel is the temperature of the telescope.

R is the mean value of the bisected values in the unit of micrometer revolution.

(IS) is the correction of progressive inequality (ascending) of the micrometric knob.

(WI) is the correction for the inclination of the moving eyepiece hair.

The indicators E and W give the positions of the telescope east or west. For the above magnitudes, we must note the following:

(1) The micrometric constant and the constant of the plane are used for corrections in the full extension of the Annual Report. These values were taken from the "Relazione sull'Attivita del Servizio Internationale delle Latitudini nel 1961, 1962" [Activity Report of the International Latitude Service in 1961, 1962].

(2) The gradual ascending (progressive inequality) of the micrometric knob is given in Table 6. There is no correction applied for periodic ascension. /49

TABLE 5. CONSTANT OF THE INSTRUMENT.

Station	Micrometer (half turn)	Positions Plane I	Positions Plane II
Mizusawa	$19''{,}9728 - 0''{,}00030\ T_{tel}\ (°C)$	$1''{,}320$	$1''{,}167$
Kitab	$19''{,}9173 - 0''{,}00012$ "	0,369	0,901
Carloforte	19,6358 —	1,030	1,247
Gaitherburg	19,8148 - 0,00027 "	1,009	1,035
Ukiah	19,8808 - 0,00004 "	1,138	1,057

TABLE 6. EXAMPLE (SAMPLE) OF THE ASCENDING OF THE MICROMETER KNOB FOR MIZUSAWA AND GAITHERSBURG.

Revolution	0	1	2	3	4	5	6	7	8	9
Mizusaua	–	–	$0{,}^{r}002$	$0{,}00\overset{2}{3}2$	0,00203	0,00182	0,00144	0,00112	0,0075	
Gaithesrburg	$-0{,}^{r}0048$	-0,0056	-0,0096	-0,124	$0{,}^{r}0171$	0,0167	0,0183	0,0215	0,0239	

(3) The spherical correction is computed for every star by the formula:

$$C = \frac{15^2}{2} \sin 1''. F^2. \tan \delta_t$$

where F is the isometric distance of the measured point from the meridian in seconds. This is ±20^s and ±6 2/3, and they are taken for all the numbers.

(4) The correction for the differential diffraction was computed for every star by the formula:

$$ref = 60''.154 \frac{1}{1+0.00367 Tex} \cdot \frac{B}{760} \sec^2 Z \cdot \Delta Z \sin 1''$$

where 60".154 is the constant diffraction.

Tex is the temperature in degrees of a hundred degree scale during the time of observation.

B is the atmospheric pressure in mmHg during the time of observation reduced to 0°C.

Z is the mean zenith distance of the pair which can be replaced in the application by one half the difference of the declination at the beginning of the year.

ΔZ is the difference in the observed zenith distances measured in seconds of arc.

(5) The corrections for the inclination of the eyepiece hair (WI) were applied to every single value of the micrometric reading as follows; only when they were separated, that is, when we had different values, 1, 2, 3, 4 indicate the sequence of the values. /50

Bisecting	(WI)	Bisecting	(WI)
1, 2, 3, -	$-\frac{1}{3} I_4$	1, -, 3, -	$-\frac{1}{2}(I_2+I_4)$
1, 2, -, 4	$-\frac{1}{3} I_3$	-, 2, 3, -	$-\frac{1}{2}(I_1+I_4)$
1, -, 3, 4	$-\frac{1}{3} I_2$	1, -, -, 4	$-\frac{1}{2}(I_2+I_3)$
-, 2, 3, 4	$-\frac{1}{3} I_1$	-, 2, -, 4	$-\frac{1}{2}(I_1+I_3)$
1, 2, -, -	$-\frac{1}{2}(I_3+I_4)$	-, -, 3, 4	$-\frac{1}{2}(I_1+I_2)$

The I_1, I_2, I_3, I_4 indicate the corrections of every bisected value in the reduced part of the star which are due to the inclination of the moving eyepiece hair. Each of the I_i values is obtained from the bisected values of the latitude observations, which are obtained during the 3 month period, and is given by the following formula and used for the mean for the 3 months:

$$I_i = \frac{1}{n}\left[\frac{1}{4}\sum_{j=1}^{n}(R_1j + R_2j + R_3j + R_4j) - \sum_{j=1}^{n}(R_i)j \mp \sum_{j=1}^{n}(C_{1j} - C_{2j})\right]$$

upper points: i = 1, 4 for Tel E, i = 2,3 of Tel w

lower points: i = 2, 3 for Tel E, i = 1,4 of Tel w

where $$C_{1j} - C_{2j} = \frac{15}{2}\,\frac{\sin 1''}{M}\,(F_1^2 - F_2^2)\tan\delta_j \quad , \begin{cases} F_1 = F_4 = 20^3 \\ F_2 = F_3 = (6\frac{2}{3})^3 \end{cases}$$

where i = positions of bisecting 1-4.

j = numbers of the sequence of the observed stars 1 to n.

n = total number of observations

$R_{ij} - R_{nj}$ = relative bisected value of a star j

$(R_i)_j$ = bisected value of a star j at position i

C_{1j} = spherical correction in the unit of the micrometer turn of the star j at the position i = 1 and equal to C_{4j}

C_{2j} = spherical correction in the unit of the micrometer turn of the star j at the position i = 2 equal to C_{3j}

M = micrometer constant

4.2.6. Specific Values for Each Pair of Latitude Observations /51

The individual values of the latitudes observed at Mizusawa, Kitab, Carloforte, Gaithersburg and Ukiah were computed with the method described, and it is shown in Table 7. In the same table the mean monthly pair values are also shown. In the table, the numbers 19, 20, ... indicate the numbers of the star pairs of the corresponding pair.

4.2.7. Zenith Distance Differences

The individual zenith distance differences were computed by the formula $(1/2)(Z_s - Z_n) = K$ + corrections (plane, spherical, refraction) and were summarized for every pair and every station each month.

TABLE 7. SAMPLE OF INDIVIDUAL VALUES OF THE OBSERVED LATITUDE.

			19	20	21	22	25	26
		IV	—	—	—	—	V —	—
Jan	8	39° 8′	3,″677	3,″588	3,″189	3,″086	3,″877	4,″020
	9		3,327	—	—	—		
	13							
	17		—	—	—	—	3,″544	4,″375
. . .								
Mean			3,377	3,600	3,109	3,298	3,400	3,801.

The monthly mean value of the zenith distance difference of the pair and the mean value of the group in the unit of revolution of the micrometer knob is indicated in Table 8. This table is given from $Z_s - Z_n$ or $R_E - R_W$.

TABLE 8. SAMPLE OF MEASURED ZENITH DISTANCE DIFFERENCES (IN MICROMETRIC TURNS).

January 1962 /52

		19	20	21	..... Mean		25	26
	IV					V		
M		$-14^{r},67$	$-18^{r},01$	$4^{r},80$	$-0^{r},33$		$-10^{r},41$	$-8^{r},88$
K		-14,84	-18,20	4,70	-0,45		-10,57	-9,05
C		-14,65	-18,03	5,18	-0,04		-10,30	-8,75
G		-14,26	-17,62	5,37	0,21		-9,96	-8,42
U		-14,28	-17,63	5,29	0,14		-10,00	-8,47.

4.2.8. Mean Latitude Values of a Group and Their Mean Epoch

The mean latitude value of a group was taken from the monthly mean pair values which result from the individual (specific) values of the latitudes as they are indicated in Table 7. This derivation is shown in Table 9. The mean epoch of the observations in 1 month was computed as the mean values of the time of all observations and was measured in units of tropical years from the beginning of the Bessel year. These are indicated in Table 9 with the mean latitude value of the group and the number of observed star pairs. Note that the "evening," "morning" and "intermediate" values are indicated separately.

TABLE 9. SAMPLE OF HOW TO FIND THE MEAN EPOCH OF OBSERVATION, MEAN LATITUDE VALUE OF A GROUP AND NUMBER OF OBSERVED PAIRS.

Mizusawa (39° 8')

e (evening)			i (intermediate)			m (morning)
Bersselian year	φ	n	Bersselian year	φ	n	
1962,067	3",338	35	1962,066	3",390	32	
,152	3",375	48	154	3,225		
,233	3",072	63	236			
,314	3,123	71				

4.2.9. Mean Latitude Values of a Group Reduced to a Common Epoch /53

Every latitude station has 12 mean epochs of observation in a year. Each of them is very close to each mean date of the programmed duration of the month, but among them they are different. In order to compute the coordinates of the pole, 12 common epochs were devised at the five I.L.S. stations, as is shown in the first column of Table 10. These were chosen so that they are very close to each mean date of the month of observation, by dividing the year into 12 equal parts.

The common epochs are invented for every year by a rule. The $0^{\tau}U^{\tau}$ on January 21, 1965, was indicated as the first common epoch in 1 year, and was given in units of the tropical year measured from the beginning of the Bessel year. The mean values of the latitude of a group and the mean epochs referred to them are indicated in Table 9 and are reduced to the closest common epochs by linear interpolation or extension between the two successive mean latitudes of the same group. The mean latitudes of a group in the mean epochs are indicated in Table 10.

TABLE 10. SAMPLE OF THE MEAN LATITUDE VALUE REDUCED TO THE MEAN EPOCH (39°8').

Bessellian Year	Group	Mizusawa φ	n	Kitab φ	n	Carloforte		Gaithesrburg	
1962,056	IV (e)	3″342	35	1″456	7	8″883	87	13″439	76
	V (i)	3,392	32	1,625	6	8,921	81	13,467	75
	VI (m)	3,118	35	1,276	6	8,747	57	13,284	76
1962,139	V (e)	3,377	48	1,600	49	8,908	94		
	VI (i)	3,209	36	1,440	36	8,753			
	VII (m)	3,270	23	1,405	30				

We summarize what was said in the previous two chapters: /54

(1) With corrections, we obtain the individual latitude values for every pair of each group at a defined date of the corresponding epoch (magnitude as the month).

(2) From the individual latitude values we take the mean values of each pair. Each group is already referred to one mean value of the epoch (approximately equal to that of the month).

(3) From the mean values of each pair we obtain (again with the average) the mean latitude value. The group is again referred to the mean value of the epoch.

(4) We reduce the mean latitude value of each group, which is referred to one mean value of the corresponding epoch, to the corresponding common epoch, common for the five I.L.S. stations.

We will apply the above together for Mizusawa, for the pair of Group IV for the epoch January 8 - February 5.

INDIVIDUAL LATITUDE VALUES.

	19	20	21	22	23	24
IV						
Jan 8	–	–	–	–	–	–
9	3",677	3",588	3",189	3",086	3",672	2,781
13	3,327	–	–	–	–	–
17	–	–	–	–	–	–
23	3,351	3,541	3,456	2,914	–	–
25	–	3,929	–	–	4,276	–
27	3,475	4,199	3,010	–	3,389	3,312
29	2,866	3,357	–	3,632	3,594	2,900
30	3,864	3,637	3,424	3,544	–	2,877
31	–	–	–	–	–	–
1	3,096	2,948	2,467	3,315	3,513	2,892
3	3,163	–	–	–	–	–
5	–	–	–	–	–	–

/55

Mean values 3.377, 3.600, 3.109, 3.298, 3.685, 2.952 of each pair.

Mean value of the latitude of a group referred to the mean value of the epoch $^{8}/_{J} - ^{5}/_{F} = (1/6)[3.377 + 3.600 + 3.109 + 3.298 + 3.689 + 2.952] = 3''.338$.

Therefore, for the mean value of the epoch, we obtain, for 1962, 3".338 (in the unit of the tropical year from the beginning of the Bessel year. The beginning of the Besselian year is when the sun has right ascension 280° near 1/1.) The duration of the tropical year is 366.242199 sidereal days and 365.242199 mean solar days.)

Already we can reduce to the mean epoch, so we will have:

$$1962.056 \rightarrow 3.342 \text{ (for group IV)}$$

This result is obtained by linear interpolation of the results:

1962.067	3".338
1962.152	3".375

Thus we find for the value 1962 → 3".342.

Note that when we refer to a certain group, we refer to evening, morning or intermediate, respectively, because each group is observed at a special time of day.

That is, $\varphi_e = \varphi_{IV}$, $\varphi_i = \varphi_{II}$, $\varphi_m = \varphi_{III}$

so instantaneous x,ϕ corresponding to instantaneous $\phi \rightarrow x_e = x_{IV}$, ...

4.2.10. Determination of x,y and the Term Z

/56

The coordinates of the instantaneous pole and the term Z are computed with reference to the new system (1900-1905) of the I.L.S. from the results of the five I.L.S. stations; these are indicated in Table 10. The mean latitude values of the corresponding stations defined in the new system (1900-1905) and the longitude values finally taken are:

Station	λ	ϕ (1900-1905)
Mizusawa	-141° 7' 51"	39° 8' 31",602
Kitab	- 66 52 51	1,850
Carloforte	- 8 18 44	8,941
Gaithersburg	77 11 57	13,202
Ukiah	123 12 35	12,096

The latitude difference $\Delta\phi$ at a station of longitude λ is given by the formula:

$$\Delta\phi = x\cos\lambda + y\sin\lambda + Z$$

where x, y are the coordinates of the instantaneous pole and Z is the nonpolar variable. x,y and Z are computed separately for the evening, morning and intermediate observations, by the following usual formula, and the use of five different series for the five stations:

$$\Delta\phi = \phi - \Phi$$

where Φ are the mean values of the latitude of the station, and ϕ are the instantaneous values of the latitude of the station, and $\Delta\phi$ is given from the above formula. According to the above formulas, if we take the five series and solve them, we will obtain the following values:

$$x = -.4359\Delta\varphi_M + .1227\Delta\varphi_K + .4483\Delta\varphi_C + .1232\Delta\varphi_G - .2583\Delta\varphi_U$$

$$y = -2636\Delta\varphi_M - 3133\Delta\varphi_K - 0172\Delta\varphi_C + 3382\Delta\varphi_G + 2559\Delta\varphi_U$$

$$Z = +2305\Delta\varphi_M - 2007\Delta\varphi_K + 1755\Delta\varphi_C + 1850\Delta\varphi_G + 2082\Delta\varphi_U$$

where the indices M, K, C, G, U refer to the five stations. That /57 means that it is enough to find at an instant t the latitudes at the five I.L.S. stations, to take the differences $\Delta\phi = \phi - \Phi$ where Φ are the mean coordinates of the stations and to form the system of equations:

$$\Delta\phi_i = x\cos\lambda_i + y\sin\lambda_i + Z$$

Then we solve this system of equations with the least squares method and compute the values of x,y and Z at the moment t. The values of x,y and Z were computed for every month, as is indicated in Table 11, where the indices e, i and m are mean positions for the evening, morning, and intermediate, correspondingly to the values of the latitudes in Table 10.

Thus, after elaboration, we obtain the x_{mean}, y_{mean} values as is indicated in Table 11, so we can get the values of the polar coordinates for every 0Y.05. Also, we can obtain the graph of the values of the coordinates. Numerically this is indicated in Table 12.

The barocentric value of the years 1900-1905 is taken as the original point. The coordinates of the barocentric value of the orbit of the pole referred to the new system 1900-1905 were computed from the components of the pole over 6 years, from 1957-1961 and 1962 for the mean value of the date 1960. Their values are:

$$x = 0''.063 \qquad y - 0''.205$$

Below we will give a sample of the two tables 11 and 12. In Table 11 are the values of x, y and Z which are computed from the mean latitude values and the instantaneous ones, which we took finally from Table 10. Table 12 has the mean values of x, y for a portion of the year equal to 0y.05.

We will attempt a solution of the system $\Delta\phi_i = x\cos\lambda_i + y\sin\lambda_i$. using the method of least squares. /58

$\Delta\phi_i = x\cos\lambda_i + y\sin\lambda_i$ where i = 1, 2, ... 5 show the five stations. The system is of the form $a_i x + b_i y - l_i = 0$.

We form the table:

a_i	β_i	l_i	$a_i\beta_i$	$a_i a_i$	$b_i b_i$	$a_i l_i$	$B_i l_i$
$\cos\lambda_1$	$\sin\lambda_1$	$\Delta\varphi_1$	$\cos\lambda_1\sin\lambda_1$	$\cos^2\lambda_1$	$\sin^2\lambda_1$	$\Delta\varphi_1\cos\lambda_1$	$\Delta\varphi_1\sin\lambda_1$
$\cos\lambda_2$	$\sin\lambda_2$	$\Delta\varphi_2$	$\cos\lambda_2\sin\lambda_2$	$\cos^2\lambda_2$	$\sin^2\lambda_2$	$\Delta\varphi_2\cos\lambda_2$	$\Delta\varphi_2\sin\lambda_2$
$\cos\lambda_3$	$\sin\lambda_3$	$\Delta\varphi_3$	$\cos\lambda_3\sin\lambda_3$	$\cos^2\lambda_3$	$\sin^2\lambda_3$	$\Delta\varphi_3\cos\lambda_3$	$\Delta\varphi_3\sin\lambda_3$
$\cos\lambda_4$	$\sin\lambda_4$	$\Delta\varphi_4$	$\cos\lambda_4\sin\lambda_4$	$\cos^2\lambda_4$	$\sin^2\lambda_4$	$\Delta\varphi_4\cos\lambda_4$	$\Delta\varphi_4\sin\lambda_4$
$\cos\lambda_5$	$\sin\lambda_5$	$\Delta\varphi_5$	$\cos\lambda_5\sin\lambda_5$	$\cos^2\lambda_5$	$\sin^2\lambda_5$	$\Delta\varphi_5\cos\lambda_5$	$\Delta\varphi_5\sin\lambda_5$
			[ab]	[aa]	[BB]	[al]	[Bl]

Normal Equations

$$\sum_{i=1}^{5}\cos^2\lambda_i\, x + \sum_{i=1}^{5}\sin\lambda_i\cos\lambda_i\, y = \sum_{i=1}^{5}\Delta\varphi_i\cos\lambda_i \qquad (1)$$

$$\sum_{i=1}^{5}\sin\lambda_i\cos\lambda_i\, x + \sum_{i=1}^{5}\sin^2\lambda_i\, y = \sum_{i=1}^{5}\Delta\varphi_i\sin\lambda_i \qquad (2)$$

The system of equations (1), (2) will be solved by the method of determinants.

If it is

$$\begin{aligned} \alpha_1 x + \beta_1 y &= \gamma_1 \\ \alpha_2 x + \beta_2 y &= \gamma_2 \end{aligned} \Rightarrow x = \frac{\begin{vmatrix} \gamma_1 & \beta_1 \\ \gamma_2 & \beta_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}}$$

Then

$$x = \frac{\begin{vmatrix} \Sigma\Delta\varphi_i \cos\lambda_i & \Sigma\sin\lambda_i\cos\lambda_i \\ \Sigma\Delta\varphi_i \sin\lambda_i & \Sigma\sin^2\lambda_i \end{vmatrix}}{\begin{vmatrix} \Sigma\cos^2\lambda_i & \Sigma\sin\lambda_i\cos\lambda_i \\ \Sigma\sin\lambda_i\cos\lambda_i & \Sigma\sin^2\lambda_i \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} \Sigma\cos^2\lambda_i & \Sigma\Delta\varphi_i \cos\lambda_i \\ \Sigma\sin\lambda_i\cos\lambda_i & \Sigma\Delta\varphi_i\sin\lambda_i \end{vmatrix}}{\begin{vmatrix} \Sigma\cos^2\lambda_i & \Sigma\sin\lambda_i\cos\lambda_i \\ \Sigma\sin\lambda_i\cos\lambda_i & \Sigma\sin^2\lambda_i \end{vmatrix}}$$

We will compute the determinant of the denominator: /59

$$\begin{vmatrix} \Sigma\cos^2\lambda_i & \Sigma\sin\lambda_i\cos\lambda_i \\ \Sigma\sin\lambda_i\cos\lambda_i & \Sigma\sin^2\lambda_i \end{vmatrix} = \sum_{i=1}^{5}\cos^2\lambda_i\sin^2\lambda_i + \sum_{j=1}^{5}\cos^2\lambda_j\Big(\sum_{i=1}^{5}\sin^2\lambda_k\Big) - \sum_{i=1}^{5}\cos^2\lambda_i\sin^2\lambda_i -$$

$$- 2\sum_{j,k=1}^{5}\sin\lambda_j\cos\lambda_j\sin\lambda_k\cos\lambda_k = (\text{here } j \neq k)$$

$$= \sum_{j=1}^{5}\cos^2\lambda_j\Big(\sum_{k=1}^{5}\sin^2\lambda_k\Big) - 2\sum_{j,k=1}^{5}\sin\lambda_j\cos\lambda_j\sin\lambda_k\cos\lambda_k =$$

$$= \left\{\sum_{\substack{j,k=1 \\ j\neq k}}^{5}(\cos_j\sin_k - \cos_k\sin_j)^2\right\} \Rightarrow \left\{\sum_{\substack{j,k=1 \\ j<k}}^{5}\sin^2(j-k)\right\}$$

We will compute the determinant of the numbers:

$$\begin{vmatrix} \Sigma\Delta\varphi_i \cos\lambda_i & \Sigma\sin\lambda_i\cos\lambda_i \\ \Sigma\Delta\varphi_i\sin\lambda_i & \Sigma\sin^2\lambda_i \end{vmatrix} = \sum_{i=1}^{5} \Delta\varphi_i \cos\lambda_i\sin^2\lambda_i + \sum_{i=1}^{5} \Delta\varphi_i\cos\lambda_i \left(\sum_{j=1}^{5}\sin^2\lambda_j\right) -$$

$$- \sum_{i=1}^{5} \sin^2\lambda_i\,\Delta\varphi_i\cos\lambda_i - \sum_{i=1}^{5} \Delta\varphi_i\sin\lambda_i\left(\sum_{j=1}^{5}\sin\lambda_j\cos\lambda_j\right)$$

here $i \neq j$

$$= \sum_{i=1}^{5} \Delta\varphi_i \cos\lambda_i \left(\sum_{j=2}^{5}\sin^2\lambda_j\right) - \sum_{i=1}^{5}\Delta\varphi_i\sin\lambda_i\left(\sum_{j=1}^{5}\sin\lambda_i\cos\lambda_j\right) =$$

$$= \sum_{\substack{j=1\\ j\neq 1}}^{5} \Delta\varphi_1\sin\lambda_j\,(\cos\lambda_1\sin\lambda_j - \sin\lambda_1\cos\lambda_j) + \sum_{\substack{j=1\\ j\neq 2}}^{5} \Delta\varphi_2\sin\lambda_j\,(\cos\lambda_2\sin\lambda_j - \sin\lambda_2\cos\lambda_j) +$$

$$+ \cdots + \sum_{\substack{j=1\\ j\neq 5}}^{5} = \sum_{\substack{i,j=1\\ j > i}}^{5} \Delta\varphi_i \sin\lambda_j \sin(\lambda_j - \lambda_i)$$

In the same manner, if we work for the determinant of the nominator of y:

$$\begin{vmatrix} \Sigma\cos^2\lambda_i & \Sigma\Delta\varphi_i\cos\lambda_i \\ \Sigma\sin\lambda_i\cos\lambda_i & \Sigma\Delta\varphi_i\sin\lambda_i \end{vmatrix} = \sum_{\substack{i,j=1\\ j>i}}^{5} \Delta\varphi_i \cos\lambda_j \sin(\lambda_j - \lambda_i)$$

From the above solution, we observe that the values of x, y will be first degree functions of the values of $\Delta\phi_i$, which are the differences between the mean and instantaneous coordinates of the latitude ϕ of the place of observation.

Generally, the system can be solved with the same method if we take five equations with three unknowns, if the computation of Kimura's term is needed. In the same manner, we make a table of the coefficients of the unknown quantities of the observation equations so that we will obtain the normal equations: /60

$$
\begin{aligned}
[\cos^2 \lambda_i]x + [\cos\lambda_i \sin\lambda_i]y + [\cos\lambda_i]z &= [\cos\lambda_i \Delta\varphi_i] \\
[\cos\lambda_i \sin\lambda_i]x + [\sin^2\lambda_i]y + [\sin\lambda_i]z &= [\sin\lambda_i \Delta\varphi] \\
[\cos\lambda_i]x + [\sin\lambda_i]y + 5z &= [\Delta\varphi]
\end{aligned}
$$

the solution of which will give the values of x, y, Z.

In this case, we do not have the normality of the previous formulas, and the formulas are rather complicated.

4.2.11. Corrections of Declination

The declination corrections of a pair of stars are expressed in two parts. The first is called the "correction of the mean value of the group" and the second "correction of the group."

These can be defined at the same time as the error of the proper motion from the results of the latitude observations when we have enough information from many years.

At the present time, however, the information available is so poor that the error due to proper motion is ignored.

(I) Correction of the Mean Value of the Group

It could be considered as the best method of defining the correction of the mean value of a group to use only the results of all the evening observations, when all pairs which belong to the same group are observed. The difference between the value of the latitude for every separate pair and the mean value of all the pairs which belong to a group will give the "correction of the mean value of a group." /61

At a few stations like Mizusawa, the weather conditions are so unstable that full observations in one night are rare. The following method is the one used by almost all the stations.

At first, the monthly mean values of latitude are defined for each pair as:

$$\phi_i t_0 = \frac{1}{n_i} \Sigma \varphi_i t$$

where $\phi_i t_0$ is the monthly mean value of the latitude of the pair i at the mean epoch t_0.

$\phi_i t$ is the individual latitude of the pair i at the epoch t of a month.

n_i is the number of observations in one month for the pair i.

These are given for every month on the last line of Table 7. After these, the mean value of the latitude of a group is computed by computing the $\phi_i t_o$ which belongs to the same group.

$$\varphi_k t_o = \frac{1}{6} \Sigma \varphi_i t_o \; ; \; \left(\frac{1}{6}\right)$$

(1/6 because each group has six pairs) where $\phi_k t_o$ is the mean value of the latitude of a group, more specifically, of the group k at the mean epoch t_o where the i pair belongs. These are given in Table 9. (All the above are described in detail in Section 4.2.9, where a numerical example is also given.) Then the correction for the mean value of a group of the i pair will be:

$$r_i = \phi_{kt_o} - \phi_{1t_o} \qquad [\phi_{it} \rightarrow \phi_{1t} \rightarrow \phi_{kt_o}]$$

The individual values of r_i for the evening, intermediate, and morning observations are indicated in Table 13, along with the number of observations for every month n, the mean epoch t_o, and the mean latitude value $\phi_n t_o$. The mean value of r_1 for every pair is written on the last line of the column for every pair. It should be noted that the difference between Tables 10 and 13 is that according to Table 10 we have reduced in a common epoch, while according to Table 13, we are in mean epochs. /62

TABLE 13. SAMPLE OF CORRECTIONS OF THE MEAN VALUES OF A GROUP.

Unit μ = 0".001.

				1 (Pair)		2		3	
1 (Group)				r	n	r	n	r	n
Evening	M	1962,795	3",367	-383	10	-182	11	233	11
(Oct)	K	8,06	1",687	-375	10	-325	8	259	8
	.....								
Intermed.	M	1962,733	3",349	-403	7	-72	14		
(Sebt)	K	728	1",685	-413	16				

The table is done for all the groups.

(II) Correction of a Group

Each value of Z (Kimura) is defined from the five I.L.S. stations together, and it is assumed to consist of the error of declination of the group which is observed as morning, intermediate, or evening group, repeatedly during 3 successive months. In other words, the (-Z) can be considered as the preliminary correction of declination which is applied to a group. The Z values for every group and every epoch, in the year 1962, which are given in Table 11, are arranged again, and their sign changes as is indicated in the upper half of Table 14. The preliminary corrections of a group were taken, as is indicated in the column "Mean (-Z)" in the lower half of the same table. /63
The individual values of (-Z) for the years 1960, 1961 were taken from the "Relazione" by G. Ceechin.

TABLE 14.1. TABLE OF THE -Z VALUES FOR THE YEAR 1962.

Gr Group	e	i	m	Mean	(e, i, m) - mean e	i	m
IV	."049	.018	-.006	."020	"029	-".002	-".026
V	."023	-.008	-.003	-.005	."028	-".003	-".025
...	....	....	....	....	....	....	.
Mean	.148	.134	.118	.134	.014	.001	-.015

TABLE 14.2. SAMPLE OF THE CORRECTION OF A GROUP.

Gr Group	1960 e	i	m	1961 e	i	m	1962 e	i	m	Mean (-Z)
IV	."070	."043	"023	"029	"038	"026	."049	."018	-."006	"032
V	.009	.020	.024	"014	-."003	-."011	."023	-."008	-."30	"004
...	...	...	...	...	...	...	...	...	...	...
Mean	.159	.143	.140	.147	.138	.118	.148	.34	.118	"138

(III) Correction of the Declination for Every Pair

Initially, the correction of declination for every pair was carried out by adding the two above corrections (see Table 15).

TABLE 15. SAMPLE OF CORRECTION OF THE DECLINATION FOR EVERY PAIR.

Group	Pair	$\Delta\delta$
I	1	$-.''238$
	2	$-.007$
	.	...
	.	...
	.	...
	6	$.215$

4.2.12. Latitude Corrected of Error of Declination and Remaining Latitudes /64

Every latitude observation which is given in Table 7 includes the error of declination for the five I.L.S. stations and the values in Table 15 were adopted as preliminary corrections of declination.

The monthly mean values of latitude were derived as arithmetic means of the corresponding mean epochs, from the individual means of latitude corrected for error of declination (see 4.2.9). These are indicated numerically in Table 16 and on the graph of the figure. On the other hand, the normal latitudes of epochs corresponding to the mean epochs for the relevant stations were computed by the formula:

$$\phi_{nor} = \phi + x.\cos\lambda + y.\sin\lambda$$

where Φ, λ are the mean latitude and longitude values of the station and x,y are the coordinates of the pole for the epoch obtained by the calculation formula (Chapter 4.2.10). These are given in the same table. The differences between these two are given in Table 16.

TABLE 16. SAMPLE OF HOW TO FIND THE RESIDUALS FOR MIZUSAWA.

Bessel Year	φ_{cor}	φ_{nor}	Res
1962.068	3″.375	3″.395	-0″.120
.149	3″.422	3,370	0,052
.233	3″.317	3,349	-0,032

ϕ_{cor} = latitudes corrected for error of declination

$\varphi_{nor} = \Phi + x\cos\lambda + y\sin\lambda.$

Res = residuals

(That is, the latitudes reduced from the mean values Φ and the corrected ones derive the differences Res.)

4.2.13. Final Summation /65

The differences between two successive mean values of the latitudes of a group for every station, for each month of 1962, were derived from the mean values of the latitude of a group in Table 10.

These are indicated in columns e-i, i-m of Table 17. The summation for every 12 values in columns e-i or i-m gives the final summation of these sequences. The mean value from the final two summations for each station is given on the last line. The last column, indicated as Mean, gives the mean values as a set.

Summarizing the previous chapters, we can say that we obtain a second order approximation of the latitude values. This means that we will obtain the ϕ_{nor} (mean) from the already known values of the polar coordinates x,y and the nonpolar term. However, if the Z values are arranged and if we apply the corrections again with the new values, we will obtain the new ϕ_{cor} (correction) values.

Month	Mizusawa		Kitab		. . . .		Mean	
1962	e - i	i - m	e - i	i - m	. . .	. . .	e-i	i-m
Jan.	-.052	.256	-.182	.300	. . .	. . .	-	-
Feb.	.150	-.048	.164	.033	. . .	. . .	-	-
Mar.	-.089	-.027	-.061	.017	. . .	. . .	-	-
. . . .	. . .	. . . .	. . . .	. . . .	. . .	. . .	. . .	. . .
Total	-.106	-."217	-."206	-."152			-."175*	-."199*
	-."162		-."179				-."187*	

*Mean values of the neighboring values.

4.2.14. Corrections of a Group with the Chain Method /66

The "correction of a group" for each group was assumed to be made up of two parts (4.2.11). The first part is the correction due to the error of declination of a group and the second part is the correction due to the nonpolar variable of latitude which (variable) is the same in the five I.L.S. stations. It is not always necessary to distinguish them for the purpose of defining the polar coordinates from the five I.L.S. stations.

At any rate, it would be important (very useful) to define the exact value of the correction for declination when the results of the five I.L.S. stations are combined with the ones of the independent stations, where we observe the various stars or the pair of stars using different methods and instruments. It will also be important to see the results of research on the nonpolar variable of the latitude.

In this paper, the results of the chain method, in which the final result is distributed equally among the combinations of the group and the whole summation of the errors of declination for the mean values of the group were assumed to be equal to zero.

The error of proper motion was not considered either. The results are indicated in Table 18, where the mean values of the corrections of a group are shown which are taken for the corresponding stations (the five I.L.S. stations).

The corrections of the group by the chain method were compared with the ones by the (-2) and the differences between them, of an almost yearly change by 0".03 in arithmetic value, are indicated in the same table.

TABLE 18. SAMPLE OF CORRECTION OF A GROUP BY THE CHAIN METHOD. /67

Group	$\Delta\delta$	-2	Difference
I	".013	".137	-".124
II	.197	.315	-.118
III	.090	.195	-.105

4.2.15. Checking of Latitude

The checked latitude ϕ_{ig} of station i for group g was proposed by Wm. Markowith. This is computed by the formula:

$$\varphi_{ig} = \frac{\Sigma w_p \cdot \varphi_{ip}}{\Sigma W_p}$$

where ϕ_{ip} are the monthly mean values of latitude observed at station i for the pair p, given in Table 7. W_p is the weight which is given to the pair p and is considered as W_n or W_s for every north or south pair.

The weight W_p is computed by α with consideration of W_n or W_s.

$$\frac{W_n}{W_s} = -\frac{\Sigma(Z.D.)s}{\Sigma(Z.D.)n}$$

where $\Sigma(Z.D.)_s$ and $\Sigma(Z.D.)_n$ are the summations of the zenith distances of north or south pairs of stars of one group on which they can be computed for the mean epoch of the observation.

Then the barocentric zenith distance of the group disappears.

The checked mean latitudes of a group are computed for every station as is indicated in Table 19 from the numbers of observations and the means of the epochs. The results are reduced

to the common epoch (see Table 20). The polar coordinates for the checked latitude are computed by formula 14 (Table 21, upper half), where the index c indicates the results from the checked latitude. The lower half of Table 21 gives the comparison between the results of the checked latitude and those of the usual latitude (Table 10). Their difference is very small, but it /68
becomes 0".008 or 0".009 when the zenith distance of the group becomes large. This introduces the existence of a small error in the accepted smaller value.

TABLE 19. SAMPLE OF MEAN VALUE OF CHECKED LATITUDE AND MEAN EPOCH OF A GROUP.

Mizusawa Epoch

e			i			m		
Bessel Year	φ	η	Bessel Year	φ	η	Bessel Year	φ	η
1962.067	3".339	35	1962.066	3".394	32	1962.071	3".132	35
.152	.379	48	.154	.222	36	.136	.281	23
.233	.069	63	.236	.170	47	.230	.176	38

TABLE 20. SAMPLE OF MEAN VALUE OF CHECKED LATITUDE OF A GROUP AT A MEAN EPOCH (SEE TABLE 10).

Bessel Year	Group	Mizusawa (39° 8')		Kitab (39° 8')		
		φ_c	η	φ_c	η	
	IV	3".342	35	1".459	7	
1962.056	V	.396	32	.124	6	
	VI	.116	35	.260	6	

TABLE 21.

Upper half

Lower half

It is important to note that the formula which gives the weight for the values of the difference of a group has changed. It is:

$$W = \frac{n_1 \cdot n_2}{10(n_1 + n_2)}$$

where n_1, n_2 are the numbers of observations of two successive (pairs) of a group.

4.3. Chapter 2. Results of the Latitude Observations at the Independent Stations in 1962 /69

At the present time, 32 stations, five of which are I.L.S. stations, are cooperating with the IPMS. Among them are 22 stations and two I.L.S., Mizusawa and Kitab, which are working independently on latitude observations with 27 instruments, and 18 stations doing time observations only, or doing them with latitude observations. These stations worked independently according to their own programs, and they applied relevant methods for correction of declination. Therefore, it is possible that there exist certain disagreements between the mean daily values of a few stations and the ones of the IPMS. For these stations too we work by using the methods mentioned before, that is, we compute daily or weekly mean latitude values and then monthly mean values for the mean epoch. The difference is that in this case we take into consideration the weight of the number of observations. Another difference is also that for certain instruments (VZT, FZT), the mean epoch is not measured from the beginning of the Besselian year in units of tropical years, but is taken as the centrovaric mean value of the astronomic data for every observation.

The remaining latitudes are computed as the differences of the monthly mean values of the observed latitudes and the normal latitudes at the opposite mean epoch with reference to the polar coordinates which are determined from the five I.L.S. stations in the new system 1900-1905. The normal latitudes are computed from the formula:

$$\varphi_{\text{nor}} = \Phi + x\cos\Lambda + y\sin\Lambda.$$

Since the mean latitudes were accepted at the beginning at the Central Office independently from the 1900-1905 system, certain corrections have to be applied.

4.4. Observations

/70

In Chapters 4.2 and 4.3 we saw analytically the corrections which we apply to the observed data and we also saw how we obtain the final values for the five I.L.S. stations and for the independent stations. Summarizing the above, we can separate them as follows:

1. We choose a star catalogue which we can use, and we compute the mean coordinates α_t, δ_t for an epoch according to the known data. Then we examine the atmospheric terms which have to be taken into consideration.

2. We give the known formula of reduction:

$$\varphi = \frac{1}{2}(z_s - z_n) + \frac{1}{2}(\delta_s + \delta_n)$$

and we compute the apparent declination δ (making detailed computations), and we should thus obtain δ_r after the computation of $(1/2)(z_s - z_n)$, taking the various corrections into consideration (plan. sph. refraction) (Table 8). We can find the individual values of latitude now by applying the above formula. From these (a numerical example is also given), we find the mean values for each pair, and then for each group (as average mean) for a mean epoch. After this, we reduce our results to a common epoch.

3. By the least squares method, we compute the polar coordinates x, y and the term Z.

4. Finally, we apply another correction to the declinations for two errors, and we define the ϕ_{cor}, ϕ_{nor} and the checked ... or centrovaric latitude of the stations. (This is because earlier the difference in zenith distances was corrected, and only the declination was not totally corrected.)

5. We refer to the independent stations.

In order to assume the study of the corrections to be complete, the Annual Reports for the years 1963, 64, 65, 66, 67, 68 were investigated. The description of the methods was published in the Annual Report of 1962-63. As for the rest, we did not find any differences in methods, and in the Annual Reports for the years 1966, 1967 the new CIO system was used. We also see that the number of independent stations continues to increase, as well as the variety of instruments of observation. /71

Before we finish this chapter on the detailed discussion of the methods of correction and reduction of the observations, we will give as an example the coordinates of the centrovaric term of the orbits described by the pole.

	1962	1963	1964	1965	1966 (CIO)	1967 (CIO)	1968 (CIO)
X_{bar}	0".063	0".057	0".052	0".044	0".039	0".034	
Y_{bar}	0".205	0".203	0".205	0".211	0".222	0".225	

In the errors which follow, the orbit of the pole is given during the various years in the new system 1900-1905 and in the CIO system. Various graphs of the relative corrections and the positions of the observatories in the ϕ, λ system are also given.

Also, for comparison, tables will be given which show the mean values of the polar coordinates and the mean values of Z for the mean latitude of a group and the checked latitude.

For the mean latitude of a group, the polar coordinates will be given for an increase of 0.05 year.

Also, for comparison of the values which are given by the "Monthly Notes of the International Polar Motion Service" and the ones of the "Annual Report," we give the new polar coordinates for 0".05 for 1962. /72

	X_{MH}	Y_{MH}	X_{AR}	Y_{AP}
1962. 00				
05	– 11	297	-009	297
10	4	312	008	309
15	21	319	27	314
20	42	319	47	312
25	67	313	71	304
30	92	299	95	290
35	117	280	120	271
40	142	254	144	246
45	160	221	162	214
50	169	178	173	175
55	170	132	171	132
60	160	97	157	092
65	139	75	128	068
70	109	64	094	060
75	71	65	056	067
80	28	76	017	083
85	– 12	97	-019	104
90	– 50	123	-055	128
95	– 82	158	-086	160.

x,y COORDINATES FOR THE MEAN LATITUDE OF A GROUP.

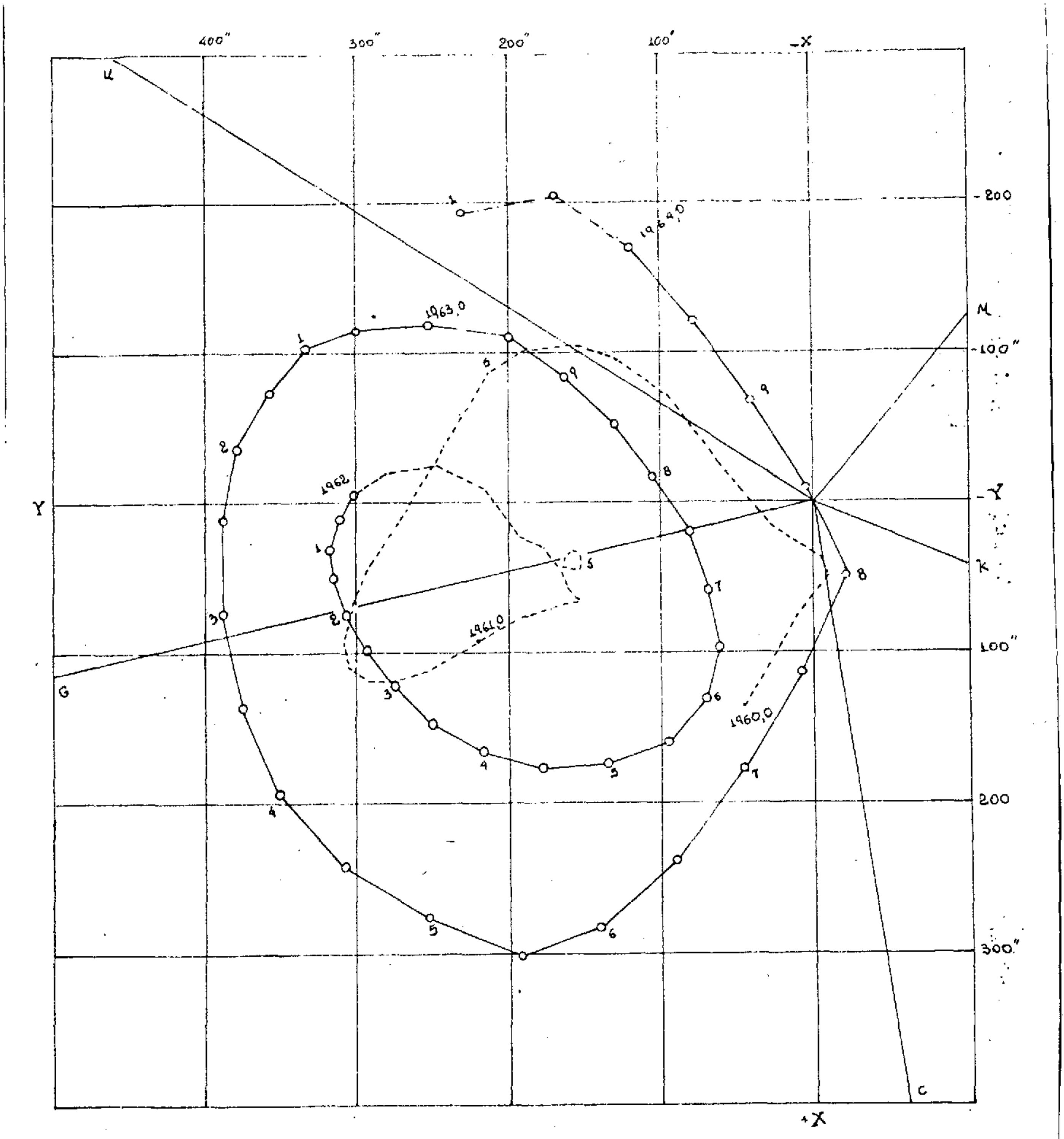
Besselian year	1962		1963		1964		1965		1966		1967	
.056	.″018	331	1-121	314	-″212	165	-075	073	047	122	068	217
.139	024	289	-083	343	-169	291	-161	133	003	100	048	166
.222	092	289	-025	377	-131	365	-182	257	-092	102	-014	144
.306	118	263	075	389	-048	475	-204	344	-098	173	010	181
.389	171	231	180	358	046	450	-136	391	-130	212	-013	130
.472	161	156	257	286	199	429	-046	448	-098	257	019	163
.556	164	087	304	178	244	328	085	435	-099	344	069	174
.639	101	054	253	106	233	227	158	358	-024	339	018	202
.722	044	084	147	026	260	153	212	283	075	326	003	206
.806	-021	102	041	-028	220	074	225	215	126	306	-030	228
.889	-082	149	-054	045	097	049	215	178	121	275	-068	255
.972	-118	219	-149	093	029	040	061	100	110	246	-063	317

x_c, y_c COORDINATES FOR THE CHECKED LATITUDE.

	1962		1963		1964		1965		1966		1967	
056	017	331	-123	317	-217	164	-081	063	042	116	078	228
139	021	290	-091	345	-181	289	-176	123	-005	089	062	176
222	088	291	-025	381	-140	367	-198	248	-109	103	-004	153
306	115	265	075	390	-061	476	-207	332	-117	164	036	190
389	168	233	180	361	043	449	-142	383	-132	209	-004	171
472	163	153	257	284	201	428	-041	447	-094	259	023	168
556	167	090	307	177	253	334	094	445	-091	350	060	166
639	104	055	261	103	250	238	164	372	-011	346	024	193
722	043	079	162	032	271	165	221	292	044	343	012	196
806	-17	094	653	-030	236	091	241	238	137	318	-031	219
889	-77	140	-042	045	097	059	236	173	140	288	-074	236
972	-117	217	-148	089	033	041	063	107	110	249	-063	308

$-0{,}04 < x_c - x < 0{,}04$

$-0{,}09 < y_c - y < 0{,}03$



Orbit of the pole (1962-1964).

A "wobble" is observed in the orbit.

KIMURA TERMS Z.

Group	1962		1963		1964		1965		1966		1967	
	Z	Z_c										
4	-.020	-.016	-132	-035	-122	001	-154	003	-113	-008	-264	-089
5	005	007	-066	009	-133	045	-077	025	-095	015	-196	179
6	-204	-209	-104	-183	-082	-204	-082	-248	-080	-248	-096	-276
7	-145	-143	+035	-128	032	-151	+044	-113	055	-139	038	-111
8	-149	-167	-182	-185	-175	-191	-177	-165	-148	-202	-188	-020
9	-028	-014	-240	006	-253	009	-228	038	-227	071	-378	-045
10	-185	-193	-029	-189	-045	-204	-003	-172	017	-159	020	-201
11	-232	-216	-172	-215	-113	-200	-137	-190	-131	-163	-064	-361
12	-005	016	-334	011	-309	063	-318	077	-346	118	-125	074
1	-137	-117	-169	-125	-201	-084	-237	-056	-214	-034	-383	-130
2	-315	282	-055	-297	-006	-296	-002	-306	-021	-308	-075	-170
3	-195	-156	-007	-171	092	-152	+022	-166	-033	-151	175	-409

5. METHODS OF OBSERVATION /76

As we already mentioned, two instruments have been used up to now for all the observations.

These were the zenith telescope of Talcott and the impersonal astrolabe of Danjon, which introduced two different methods of observation. It is interesting to mention that the development of the methods of correction and reduction up to now has been based on the method of observation of Horrebow-Talcott, with Talcott's zenith telescope.

At any rate, the instruments have been changed. Thus, for example, the photographic zenith telescope (PZT) was developed from the zenith telescope.

In what follows, we will refer briefly to the instruments and the methods of observation.

5.1. Determination of Latitude by the Horrebow-Talcott Method. Zenith Telescope

The Horrebow-Talcott method is one of the most accurate methods for determining latitude, and for this reason it is also used for the determination of its periodic changes.

According to this method, the determination of ϕ is achieved by observation of the zenith distances of two stars at their culmination within a small period of time, the one north and the other south of the zenith and approximately at the same height. As is known, the formula of computation is:

$$\varphi = \frac{1}{2}(\delta_n + \delta_s) \pm (z_n - z_s)$$

According to the Horrebow-Talcott method, the measurement of the difference in zenith distances is made only with micrometric motions, and thus the observations are free from errors which occur because of the use of arithmetic circles.

For the determination of the difference $Z_n - Z_s$, a special instrument is used which is called the zenith telescope. This instrument consists of an astronomic tube, of which the eyepiece system has a horizontal moving thread parallel to the axis of rotation, and five other threads perpendicular to the first one. The moving thread can change position parallel to itself by a micrometer knob. /77

The tube can rotate freely around a horizontal axis which is supported on a base fixed on the vertical axis. On the tube is the arithmetic circle, parallel to the optical axis. The inclination of the tube with respect to the horizontal plane is measured with the use of a vernier which is on the arithmetic circle, and with a very sensitive bubble which can be rotated around its center and can be settled on a fixed position. In order to use the instrument, its main axis is set vertical and then it is oriented. In order to observe with this instrument, we take a map of the celestial sphere and a star catalogue and choose a pair of stars which culminate within a small period of time (5-10 min) and with such zenith distances that the difference $Z_1 - Z_2$ will always be smaller than the half-diameter of the optical plane of the tube. The stars are chosen so that their zenith distances are as small as possible, and never more than 30°. In this way, we perform a catalogue of pairs of stars to be observed in the sequence in which they culminate. These catalogues have already been arranged, and the one used by the IPMS is by Boss.

In order to observe a pair, we set the zero of the vernier of the bubble in the division of the circle which corresponds to the mean value of the zenith distances $(Z_1 + Z_2)/2 = (\delta_2 + \delta_1)/2$. Then we turn the tube up to the point where the bubble takes the /78
normal position. When the first star appears in the optical plane of the telescope tube, we bisect it with the moving thread and follow it up to the moment that it crosses the meridian, which is the central vertical thread. Then we take the reading of the circle and the chronometer indications, and the ends of the bubble. Next the observation of the second star of the pair follows, which is done in the same manner, the only difference being that before this observation the telescope tube is rotated around the vertical axis 180°.

If we call λ_o the reading of the center of the bubble when the optical axis is vertical, a the angular value of the bubble deviation of the circle level, m_o the reading of the micrometer knob when the moving thread is on the optical axis, and K the value of the pace of the micrometer knob, and if we assume that the divisions of the level circle increase from one end to the other, then we will obtain:

$$Z_1 = Z_o + K(m_o - m_1) \pm a\,(\lambda_1 - \lambda_o) + R_1$$

where Z_o is the zenith distance of the point of intersection of the celestial sphere (of the star) with the optical axis of the telescope when the reading of the center of the bubble is λ_o.

R_1 is the correction for diffraction, and it is (+) or (-) if the starting position of the division of the level bubble is on the side of the eyepiece or objective lens of the telescope.

For the other star of the pair, we will have:

$$Z_2 = Z_0 + K(m_0 - m_2) \pm a(\lambda_2 - \lambda_0) + R_2$$

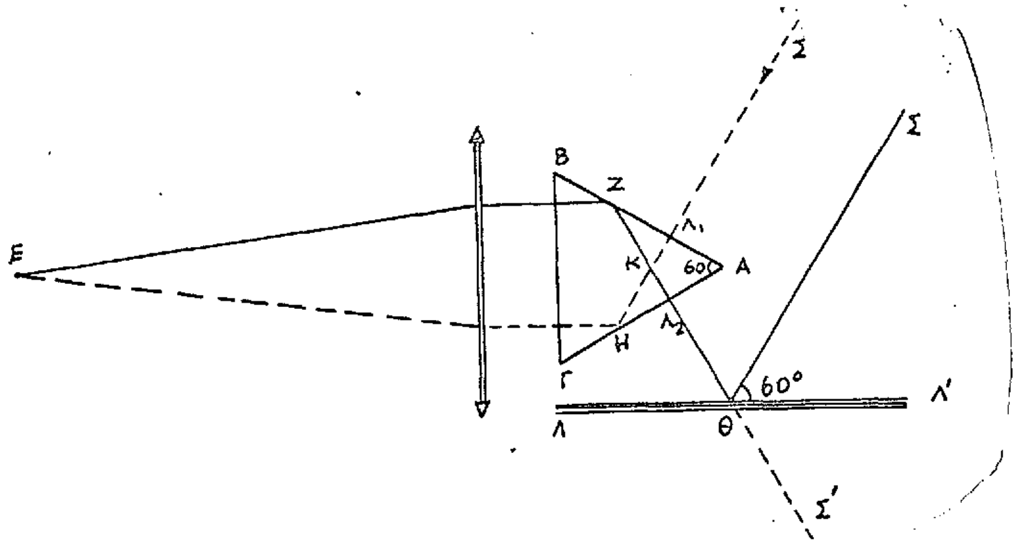
Thus $$Z_1 - Z_2 = K(m_2 - m_1) \pm a(\lambda_1 - \lambda_2) + R_1 - R_2$$

In the application by the IPMS, the formula is much more complicated, and it takes various factors into consideration, such as the temperature, the inclination of the cross thread, and others. It also considers other corrections such as plane and spherical, as has already been discussed. /79

In order to eliminate errors of the level, Cookson invented the floating zenith telescope. It is an instrument that has the base of the horizontal axis of rotation of the telescope tube floating in a basin containing mercury. In that way, the rotation of the instrument takes place around a direct vertical axis, and the rotational axis of the telescope is always horizontal. Finally, the zenith photographic tube (PZT) is used; its horizontal plane is substituted by a stable basin of mercury. The image of the star is reflected (by the Hg) on a photographic plate which rotates around a vertical axis. Many exposures are taken of the star. A special device called a "measurement device" is used for the measurements.

5.2. Prismatic Astrolabe - Impersonal Astrolabe of Danjon

The prismatic astrolabe is an instrument invented by Dlaude and Driencourt. The principle on which it is based is the following. Suppose we have an equilateral prism of which one side is vertical and all the edges of this side are vertical. This prism is set on a mercury surface which is contained in a mercury basin stably connected with the prism. The rays of a star are partly reflected by the mercury surface and partly diffracted through the prism. From the figure, we will find that if the ray ΣH is perpendicular on the edge of the equilateral prism BA, then Σθ will form a 60° angle. The ray which is reflected by the mercury surface is also diffracted through the prism and then it (total ray) intersects ΣH at a point E on the interior surface of the lens, after passing through a concentrating lens. With the change in zenith distance of the stars, the focal length of their images changes, and when the zenith distance becomes larger, the focal length becomes smaller, and /80



vice versa. We see that the angle K is 120° because $A\Lambda_1\Lambda_2K$ can be inscribed in a circle, and the angle A is equal to 60°. Thus, the zenith distance will be 30°. This is the important result which is basic for the instrument, that is if and only if the two images of the ray reflected by the mercury surface and the ray diffracted through the prism coincide, then the zenith distance of the star is 30°.

In order to make observations with an astrolabe, it must be calibrated before the start of the procedure. We must:

1. Set up the azimuth circle horizontally, with the aid of the level circle which is on the telescope tube, and the horizontal clamp.

2. Set the optical axis of the telescope horizontal. This axis is perpendicular to the rotational axis of the instrument, so we can simply set the second axis vertical.

3. Set the edges of the prism horizontal. In order to do this, we must take out the telescope eyepiece and then observe the image of a vertical thread which we hold in front of the front /81 edge of the prism of the astrolabe. If the edge is not horizontal, then its image will not be a straight line but a broken one. In this case, we correct with a special knob until we see the image as a straight line.

4. Set the rear side of the prism vertical, that is, perpendicular to the optical axis of the telescope by means of self-orientation. In this case, observing through the telescope, we attempt, by using three adjusting screws of the prism, to set the threads of the cross on their image which is the result of the reflection by the rear side of the prism. In this case, we use the eyepiece system which has a prism of total reflection, called the self-oriented eyepiece (Bohnenberger prism). This is always found with the instrument.

5. Finally, we must orient the azimuth circle of the instrument. In order to do this, we rotate the telescope so

that its optical axis becomes parallel to the direction of the magnetic needle which is attached to it, and which is corrected by a value of the magnetic declination due to daily periodic and irregular changes. Then we keep the telescope tube immovable and rotate the azimuth circle until the zero of this circle coincides with the zero index of a vernier which is attached to the telescope. Then the zero reading of the circle corresponds to the beginning of the azimuth measurements. This way, we have an approximate orientation sufficient for our observations.

The only source of undesirable errors is the case of a nonhorizontal optical axis of the telescope which must be taken into consideration. The other conditions we referred to can /82 either be fully satisfied, or can give very small errors.

In order to observe stars having equal altitudes using this instrument, we must prepare the catalogue of these stars, as is known. This catalogue contains the stars in sequence of their succession from an altitude of 60°, the sidereal times of the corresponding successions and their azimuths. The time interval between two successive observations is on the order of 2-3 min, and it varies according to the experience of the observer.

When we set the two images of the star in the optical plane of the telescope, we rotate by the azimuth, so that their coincidence will occur within the parallelogram of the threads (in order to have zenith distance error less than 0".1). At the moment of coincidence, we note the time with our chronometer, and we prepare the instrument for the next observation. The observations are quick and easily done. We must also take the temperature and pressure readings for the computation of the difference in correction of the diffraction. It is important to mention that just slightly before the coincidence of the thread we must keep the telescope absolutely immobile, so that we do not disturb the surface and make the image by reflection disappear.

The astrolabe has many advantages and disadvantages. The biggest disadvantage is that we can not use the impersonal micrometer in observations made with it. If we could, then our observations would be absolutely free of the personal equation of the observer. Research already done to solve this problem has not given any results. Usually, we try to find the personal equation by using a device; however, this method is not successful. /83

Because of the stability of the mercury surface, we can make long series of observations. However, because of changes in atmospheric conditions which cause changes in diffraction, the duration of the program of observations should not exceed 2 hours. The altitude of observation is constant, because it depends on the angle of the prism.

Another big disadvantage is the change in the mercury surface caused by the wind or by vibrations, which results in the introduction of errors. Also, just because we have only one coincidence of a star, we are not able to make many observations. For this reason, we are forced to observe many stars (more than 40). Given that the random error of each observation is quite large, we have to eliminate the errors by the least squares method of solution.

The typical instrument described above has already changed. The major change was made by Danjon, who invented the impersonal astrolabe. This is a fixed instrument, because of its weight of 180 kg, and it is used only in observatories. The principle of this instrument is the following. In the path of the two rays which come through the prism, and close to the point of coincidence, we put a double Wollaston prism of double symmetry. This prism splits the light rays in such a manner that the angle of the split rays which come from the same bundle remains the same. Obviously, we will have four bundles of which two will be more convergent than the other two. The two diverging bundles of rays are not taken into consideration. At the moment of coincidence of the two images of the astrolabe which are produced /84 from the converging bundles (they converge more if the prism is closer ‡ change of the position of the prism), if the surface of the prism on which we observe the duplication of the rays passes through the point of image coincidence, the coincidence is maintained. Given that the angle of the two converging bundles is changing because of the motion of the star we observe, we can maintain the coincidence of the two internal images by displacing the prism by a micrometric knob. This knob carries electrical contacts connected to a drum, so that the exact times that correspond to steady declinations between the two images of the astrolabe are recorded.

The mean value of one group of such recordings, always the same, corresponds to the constant altitude of the star. On the readings of the drum of the micrometric knob, the positions of the prism are noted, as well as the positions for which we do not have duplication of the Z_n images of the 24 recorded stars. The 20 are taken as central.

If V_m is the mean reading of the 20 contacts and V_o the reading when we do not have duplication, the difference $(V_o - V_m)$ changes according to the change in altitude of the observation.

This change as well as the diffraction is the only correction which can be applied to the zenith distance of 30°

in order to find the zenith distance of the observation. Because the focal plane of the telescope is inside the Wollaston prism, we can not use threads which could determine the usable part of the optical plane.

That is, by displacing the prism, since its surface passes through the point of coincidence, the coincidence is maintained, and thus by micrometric continuous displacement we have more contacts which give the $(V_m - V_o)$. /85

5.3. The Impersonal Micrometer

The impersonal micrometer is the basic device for the above-mentioned methods of observation. This is because, first of all, the measurements by micrometric motions are better than those by graduated disc, as far as random errors are concerned, and secondly, by using the impersonal micrometer, the personal equation of the observer is avoided. The only method that can be compared to that of the impersonal micrometer is the photographic one. For this method, it is important to describe and study it.

As is known, during observations we wish to know the exact time that a star crosses the vertical or the horizontal thread of the cross. But the reception of the time by electrical circuit introduces errors which depend on the operator, and that is why we refer to them as the "personal equation of the observer." In order to eliminate these errors, the impersonal micrometer was invented, that is, another cross consisting of two parallel threads a small distance apart. The system of the two parallel threads can move parallel to the multiple cross of the telescope.

If we assume that a star reaches the optical plane of the telescope, then we displace by a knob the system of vertical threads up to the point when we set the star in their center, and then, by continuous rotation, we follow the motion of the star in the optical plane systematically. /86

Since there is no chronograph connection, every time the predetermined graduations of the drum pass from the contacts, we will have a time recording. In this manner, we avoid the personal equation of the observer. It is to be noted that the drum has graduations of high precision. In the Wild T4 theodolite, one rotation of the drum corresponds to 150".

We shall call the pace of a knob K the displacement, in seconds of arc, of the movable thread (if we assume that there is one thread) of the micrometer, when the drum of the contacts makes one full rotation. This value is useful for the conversion of the micrometer readings, which correspond to the turns of the drum and portions of turn, in seconds of arc.

If we assume that we have one thread in the movable system and that we bisect a star with it, and if a_0 is the reading on the drum when the movable thread and the nonmovable thread coincide; then if a_1 is the reading of the drum for the bisecting of the star, the quantity $a_0 - a_1$ is the distance of the star from the nonmovable thread. In order to transform this quantity into seconds of arc, we obviously have to multiply with the pace of the knob K. So if a is the reading of the aiming, it will be:

$$\Sigma_1 = B_1 + K(\alpha_0 - \alpha_1).$$

In the same way, if we make other observations, we will have a system of equations of the form:

$$\Sigma i = \beta i + K(\alpha_0 - \alpha_1) \qquad \text{or} \qquad K\alpha i = \beta_i + \Sigma_i + K\alpha_0$$

Then, with the least squares solution, we determine K. In order to eliminate the dispersion of the intervals in recording, we make many turns of observations, or we use an electric motor which is set so that it gives a velocity a little smaller than that of the star, so that the observer does not have to change /87
the velocity to a smaller one, but only to a larger one. The motor-driven impersonal micrometer eliminates the dispersion to a few 0".01 instead of the 0".1 of the hand-driven impersonal micrometer.

6. MAJOR NUTATIONAL TERMS

6.1. Introduction

Closing the chapter on observations, it is useful to go back to the material of the I.L.S. which exists up to the present.

In doing this review, we shall try to find the main nutational terms. The problem will give us a more general idea of the givens [sic] and the methods of correction, and their detailed analysis will become a way of thinking for similar studies. The examination of nutation by analyzing the observations of latitude, making reductions and harmonic analysis will give the main nutational terms. This chapter was considered indispensable and the final one, because polar motion is examined more generally, the correction of declination by Talcott's method is justified, and the chapter on precession and nutation closes. Thus, the theory found in the following chapters will be better understood. It is considered necessary to given an introductory summary on nutation and to restrict the problem. Simply because the material of the given data and the corrections is very detailed, the work which follows will be given in the form of a summary. In this manner, we will avoid the danger of going off the main subject by referring to details.

Because of the influence of the Sun, the Moon and the other planets on the equinoctial swelling of the Earth is not on the /88
plane of its orbit and because of the rotation of the Earth, we will have, on the one hand, one displacement of the equinox point γ in the counterclockwise direction; on the other hand, one periodic change of the obliquity of the ecliptic, which is called nutation of the celestial axis. The precession of the equinoxes and the nutation of the celestial axis will have equations of the form:

$$(\widehat{\gamma\gamma_0}) = a_1(t-t_0) + a_2(t-t_0) + p\sin 2\lambda_S + q\sin 2\lambda_M + b\sin\Omega + c\sin\Omega + \ldots$$

$$\omega = \omega_0 + p_1\cos 2\lambda_S + q_1\cos 2\lambda_M + b_1\cos\Omega + c_1\cos\Omega + \ldots$$

where $\omega_0 = 23°27'\ 8''.26 - 468.44\, t_1 - 0''.60\, t_1^2 + 1''.83\, t_1^3$

ω_0 is the mean value of the obliquity of the ecliptic. It is given in theory that the period of p_1 is 6 months, the period of q_1 is 14 days (13.7), of b_1 18 2/3 years. Exactly these coefficients will be examined in detail.

6.2. Brief Examination

In Talcott's formula $\phi = \delta + Z$, we use, for the influence of nutation on declination, the formula:

$$\Delta_0\delta = -N_0 (\eta_0 \cos\alpha \sin\Omega - \sin\alpha \cos\Omega).$$

From the resulting errors (i.e., the differences in errors), we obtain the existence of an 18 2/3 term. We assume that these errors come from the values of N_0, n_0 (constant of nutation, ratio of axes without nutation) and the value of the right ascension. Thus the above formula becomes, because of the errors:

$$\Delta\delta = (N_0 + \Delta N)\left[(\eta_0 + \Delta N) \cos\alpha \sin(\Omega - \beta_1) \sin\alpha \cos(\Omega - \beta_2)\right]$$

So it will be

$$\varphi = Z + \delta - \Delta_0\delta + \Delta\delta \quad \text{or} \quad \varphi + \Delta\varphi = Z + \delta$$

Obviously, we need to determine the corrections of Δn, Δv,* b_1 and b_2. By appropriate transformations, we reduce the formula to: /89

$$\Delta\varphi = A_1 \cos\alpha \cos\Omega + \beta_1 \sin\alpha \cos\Omega + A_2 \cos\alpha \sin\Omega + \beta_2 \sin\alpha \sin\delta_0$$

or

$$\Delta\varphi = d_1 \cos(\Omega - \alpha) + b_1 \sin(\Omega - \alpha) + a_2 \cos(\Omega + \alpha) + b_2 \sin(\Omega + \alpha)$$

Thus it is correspondingly necessary to determine the A_1, B_1, A_2, B_2, or the a_1, b_1, d_2, b_2 to be quite small.

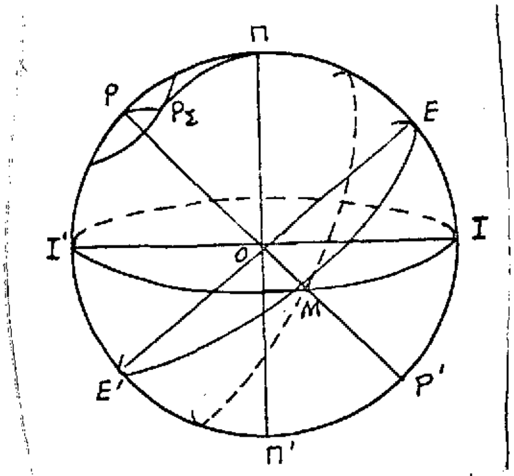
The complete analytical solution of which the harmonic investigation gives the periods is the following:

The source of the main terms which are caused by the Moon can be described as follows. We take n as the mean motion of the Moon and K the ratio of the masses of the Earth and the Moon. The factor n' is substituted by $n^2(K/HK)$.

We obtain an axis on the plane of the Moon orbit with pole P_Σ. On this frame, the direction cosines of Π are (l, m, n) (not of course the same n) those of the Moon $(\cos\mathbb{C}, \sin\mathbb{C}, 0)$. Then it is

$$\cos \Pi M = l \cos\mathbb{C} + m \sin\mathbb{C}$$

*[Translator's note: The original has a "v" or "u" here; the term should, however, be ΔN.]



and its mean value for 1 month is

$$\text{εἶναι}\ \frac{1}{2}(l^2+m^2)=\frac{1}{2}-\frac{1}{2}n^2=\frac{1}{2}-\frac{1}{2}\cos^2\Pi P_\Sigma=\frac{1}{2}\sin^2\Pi P_\Sigma$$

Now $XPP_\Sigma = -pt$ (X = position assumed by the origin), where p is the mean motion of position of the Moon on the ecliptic and $X\hat{P}P_\Sigma = \psi$ (the additional constant which can be given when necessary). Thus:

$$\cos\Pi P_\Sigma = \cos i\ \cos\vartheta + \sin i\ \sin\vartheta\ \cos(\psi + pt)$$

θ = angle of obliquity, ψ = length, W = function of work done by the Moon (dynamic investigation) and

$$W=\frac{3}{4}\ \frac{n^2K}{1+K}\ \left((C-A)\ \cos^2\Pi P_\Sigma + \text{constant}\right) = \frac{3}{4}\cdot\frac{n^2K}{1+K}\ (C-A)\Big[\cos^2 i\ \cos^2\vartheta + \frac{1}{2}\sin 2i\,\sin 2\vartheta\,\cos(\psi+pt)+\frac{1}{2}\sin^2 i\ \sin^2\vartheta\,[1+\cos 2(\psi+pt)]\Big]$$

Then /90

$$\frac{\partial W}{\partial\vartheta}=\frac{3}{4}\ \frac{n^2K}{1+K}(C-A)\Big[-\left(\frac{3}{2}\cos^2 i-\frac{1}{2}\right)\sin 2\vartheta + \sin 2i\ \cos 2\theta\ \cos(\psi+pt) + \frac{1}{2}\sin^2 i\ \sin^2 2\theta\ \cos 2(\psi+pt)\Big]$$

$$\frac{\partial W}{\partial\psi}=\frac{3}{4}\ \frac{n^2K}{1+K}\ (C-A)\Big[-\frac{1}{2}\sin 2i\ \sin 2\theta\ \sin(\psi+pt)-\sin^2 i\ \sin^2\vartheta\ \sin 2(\psi+pt)\Big]$$

The constant part of $\partial W/\partial\psi$ gives one constant elimination of ψ which is a contribution to the precession.

This is more than twice that of the Sun (2.2 times).

The part of $\partial W/\partial\theta$ and $\partial W/\partial\psi$ with an arc angle $\psi + pt$ gives a motion of a period equal to the period of rotation of the pole of the moon P_Σ, that is 18.6 years. This is the main nutation. The terms with arc $2(\psi + pt)$ cause a much smaller nutation, having a period of 9.3 years. The addition of the change of $\cos \Pi M$ within a month gives terms of a period of 1/2 lunar month proportional to the solar periods of 1/2 year, but in more complicated form. In a complete investigation, it is necessary to consider also the eccentricity of the orbits and the various disturbances of the Moon and the Sun. This has been done by Woolard. Fedorov's work is referred to the terms of angles $\Omega = -bt + \text{const}$ and $2\,☾$, and it is quite sufficient in the sense of giving a summary of how these terms arise.

As the data collected for our observations, we consider the values of the instantaneous latitude in three stations, i.e. Carloforte, Ukiah, and Mizusawa. Obviously, these values must be changed, and from these changed values, the determination of the coefficients desired will be made. /91

We consider the equation $F_1 = 0.402C + 0.302M + 0.296U$ where C, M, U are the mean values of the instantaneous latitude for a year for each pair in the three stations. This formula is similar to that of the Kimura term. So F_1 is independent of polar motion. We will try to determine the long-term change, annual and periodic. (1) Because of certain changes, we have various circles. So we require a reduction of the results of these circles to a common system. We choose the system of Boss -- GC. The correction in the declination for reduction to a common system was calculated by the formula

$$D\delta = (\delta_{GC} - \delta_{LS} + \Delta\mu\tau) + \Delta\mu\,(K - K_0).$$

where $(\delta_{GC} - \delta_{LS})$ is the difference in declination according to GC and to the results of the I.L.S., $\Delta\mu = \mu_{GC} - \mu_{LS}$ with regard to the proper motions, $(K - K_0)$ = difference between the catalogue time and 1900.00. (2) The correction for the annual component of the term τ because of a change in the date will be made by a correction f, where:

$$f = 0''.38 \sin(O_1 - O_2)\cos(O_1 + O_2 - \alpha).$$

for the latitudes after the change in date (1922.7); O_1, O_2 = values of the latitude of the Sun.

(3) The correction for the attraction due to Jupiter and Saturn is computed by a formula of the form:

$$\xi = -c\,[\sin\varphi \cdot \sin(\lambda - \alpha) + \sin\vartheta \cdot \cos\varphi - \cos\lambda]$$

where C = constant, θ = obliquity, λ = latitude of a planet.

(4) In 1912, Ross showed that not all nutational terms had /92
been taken into consideration. Their introduction, though, was done inhomogeneously (the mean latitude values before 1922.7 and the ones in space later); also, Wanach, during their introduction, got the same value for declination of Jupiter and Saturn, which is a mistake. Thus it was required that the correction n be redetermined by Uemse. The corrected values of F_i will then be:

$$F_2 = F_1 + \Delta\delta + \xi + \eta + f.$$

(5) One of the most sophisticated and demanding phases of the computations is the determination of the correction for the value of the micrometer knob, which can be found if the zenith distance of a pair is known. The zenith distance expressed in micrometric turns can be written $m = (\phi - \delta)/R$ where R = constant for all the stations, ϕ = latitude. We will determine the errors of magnitude R (R = mean value of micrometric knob) where for the three stations we obtain: $R = 0.402\ R_C + 0.30\ 2R_M + + 0.296\ R_U$.

The exact R values can not be found. We simply find certain behavioral laws. We introduce a correction of R, $\Delta R = (S_- - S_+)/(m_+ - m_-)$, where m_+ and m_- are the annual values of the zenith distances of a pair, expressed in rotations of a micrometric knob, S_+, S_- are the mean values of the quantity F_2 ((-) denotes an original point). The formula would be exact if the declinations were exact, and the proper motions had indeed, as assumed, had a linear behavior (which holds only for intervals of the change of F_2). Depending on the three circles of observation, we make some corrections of ΔR and we construct its curve of change, which /93
we examine in detail and make changes in; we then construct a table giving the final ΔR values, and thus the corrected value of F_2 will be:

$$F_3 = F_2 + m\Delta R$$

(6) We have already eliminated the sources of systematic error. In order, however, to have an harmonic analysis of F_3, and to avoid arbitrary errors, we must also eliminate the non-periodic changes of F_3. So for the groups A, B, C, (circles of observation) we compose separately for each pair the annual values of F_3 and then we subtract them from 1900. Their differences will be S_A, S_B, S_C, and we reduce the magnitudes S_B, S_C to the group A.

$$S'_B = S_B + \Delta_{AB} \quad , \quad S'_C = S_C + \Delta_{AC} \text{ where } \Delta_{AB} = S_A - S_B \text{ and } \Delta_{AC} = S_A - S_C$$

(Δ_{AB}, Δ_{AC} are the systematic differences), and we will have the value $F_4 = F_3 + \Delta s$ where Δs has an appropriate value for each group A, B, or C.

We could suppose that from the information originally given, we can directly determine the coefficients of the nutational terms. We did not, however, take into consideration the errors due to declination x, and the error due to special motion y, in which case the F_4 obtained from observations and the quantity $\Delta\phi$ (=v) will be connected by relations of the form $F_4 = x + ky + v$. We make successive approximations, neglecting at the beginning the periodic term v, and so we determine x and y.

We form F_s where $F_s = F_4 - (x_1 + ky_1)$ ($=\Delta\phi$). After proper manipulations, equating with $\Delta\phi$, we obtain the values of a_1, a_2, b_1, b_2 and so we determine the values of N, Δn, β_1, β_2 which were desired.

An interesting matter that appears is the following. We assumed R to have linear behavior, and the value of n to be increasing (correcting). These do not hold, i.e. the value of /94
n requires a correction depending on the separate values of the correction, by a dependence which appears in the values of ΔN, a term of 1/2 year. From an examination, it is not proved sufficiently that the term of 1/2 year is an error in the value of R. In other words, it is probable that a term of 1/2 year exists.

In order to determine whether the assumption of linearity affects the calculations, we form the identity

$$F_6 = X_2 + Ky_2 + K^2 Z_2$$

and we determine by least squares the value of Z_2. Finally, in order to take into consideration also the systematic errors in R, we form the formula

$$F_7 = F_4 - (x_2 + Ky_2)$$

where we substitute F_4 for F_6 so that we will have the solutions from the previous solution of the system. F_r now gives the information needed for harmonic analysis.

An important problem which has been attacked by harmonic analysis methods is whether there exist terms of periodicity of the order of 18 years (as Kimura mistakenly found a term of 11 years). We will examine the other form of analysis, which we originally gave, where we denote the frequency by μ, where $\mu = I\Pi/T$.

$$F_r = A_1 \cos\mu t \cos\alpha + B_1 \cos\mu t \sin\alpha + A_2 \sin\mu t \cos\alpha + B_2 \sin\mu t \sin\alpha$$

This can be written F_r = Acosa + Bsina.

We consider A, B fixed (examining for small periods) and we apply a special method of harmonic analysis, finally determining A, B for two circles, and then combining them. After a sequence /95 of hypotheses and computations, we conclude that linear periodic analysis gives a term of the order of 16 years, and <u>not</u> another one, which, because of a probable error of 1-2 years, obviously gives the term of an 18 2/3 year period. Notice that the error is rather due to the inexactness of the method.

We can, from the determination of A, B, determine the values of A_1, B_1, A_2, B_2 also for the two periods, and can make another computation for ΔN, Δn, β_1, β_2. We observe that these values agree with those given before.

With the above comparison, this chapter on the principal nutational terms concludes. The nutational terms of a smaller period (14 days, etc.) are given by different methods. A few corrections in these were made in Ross' introduction.

The main conclusion of the above examination of the observations is the nonexistence of another term besides that of 18 2/3 years, a result which has been completely justified. Finally, the statistical analysis shows that it is possible that a term of 1/2 year also exists. The idea of the probability gives rise to some doubts about the methodology used, for the probability of repeated arbitrary and systematic errors.

Finally, it should be noted that the material is not homogeneous, and many corrections have been made for this reason. This probably introduces errors in the course that we followed, and in the unavoidable approximations that we made. The better system of results relies on the material of observations in the future, and on the progress in observations and instruments.

(The material that has been used dates to 1934.)

6.3. The Major Nutational Terms /96

6.3.1. Research Methods, Original Information Given

In latitude determination by Talcott's method, the relation below is used, in general:

$$\phi_o = Z + \delta \tag{2.1}$$

where Z is the observed zenith distance and δ the phenomenal declination of the pair.

Keeping the principal nutational terms only, we can express the influence of nutation on declination by the following formula:

$$\Delta_o\delta = -N_o(\eta_o \cos\alpha \sin\Omega - \sin\alpha \cos\Omega) \tag{2.2}$$

If the values of the nutational constant N_o and the ratio of the axes of n_o (no nutation) have an error, and the nutation is delayed in phase, then the values of the latitude computed from (2.1) will include errors depending on the right ascension of the pair α and the longitude Ω of the position occupied by the Moon. Then in the change in latitude a 19 year nutation appears (the nutational term) of which the range of nutation and the phase can differ for pairs of different values of right ascension.

We assume moreover that the result of the principal nutational terms is expressed:

$$\Delta\delta = -(N_o+\Delta N)[(\eta_o+\Delta\eta)\cos\alpha \sin(\Omega-\beta_1) - \sin\alpha \cos(\Omega-\beta_2)] \tag{2.3}$$

where ΔN is the correction in the nutational constant, Δn the correction in the ratio of the coefficients of the principal nutational terms, and β_1, β_2 is the delay (difference) in phase. Then the nutational terms are not longer; they are included in the latitude change and are computed by

$$\varphi = Z + \delta - \Delta_o\delta + \Delta\delta \tag{2.4}$$

This is a reason for the 19 year latitude change ϕ_0 to be put in the formula /97

$$\Delta\varphi = \Delta_0\delta - \Delta\delta \tag{2.5}$$

The difference of the right-hand side of relations (2.2) and (2.3) will change as follows:

$$\Delta\varphi = A_1 \cos\alpha \cos\Omega + B_1 \sin\alpha \cos\Omega + A_2 \cos\alpha \sin\Omega + B_2 \sin\alpha \sin\Omega \tag{2.6}$$

where:

$$\left.\begin{array}{ll} A_1 = -N_0 \eta_0 \beta_1 & A_2 = N_0 \Delta\eta + \Delta N \eta_0 \\ B_1 = -\Delta N & B_2 = -N_0 \cdot \beta_2 \end{array}\right\} \tag{2.7}$$

or in the form

$$\Delta\varphi = a \cos(\Omega - \alpha) + b_1 \sin(\Omega - \alpha) + a_2 \cos(\Omega + \alpha) + b_2 \sin(\Omega + \alpha) \tag{2.8}$$

where the quantities a_1, a_2, b_1, b_2 have the values:

$$\begin{array}{ll} a_1 = -\frac{N_0}{2}(\beta_2 + \eta_0\beta_1) & a_2 = \frac{N_0}{2}(\beta_2 - \eta_0\beta_1) \\ b_1 = \frac{N_0}{2}\Delta\eta + \frac{1+\eta_0}{2}\Delta N & b_2 = \frac{N_0}{2}\Delta\eta - \frac{1-\eta_0}{2}\Delta N \end{array} \tag{2.9}$$

By solving Eqs. (2.7) and (2.9), we obtain:

$$\begin{array}{ll} \Delta N = -B_1 = b_1 - b_2 & \Delta\eta = \frac{A_2 + B_1\eta_0}{N_0} = \frac{(1-\eta_0)b_1 + (1+\eta_0)b_2}{N_0} \\ \beta_1 = -\frac{A_1}{N_0\eta_0} = -\frac{a_1 + a_2}{N_0\eta_0} & \beta_2 = -\frac{B_2}{N_0} = \frac{a_2 - a_1}{N_0} \end{array} \tag{2.10}$$

By using relation (2.10), the problem of determination of the corrections ΔN and $\Delta\eta$ and the problem of the difference of phase β_1 and β_2 reduces to the determination of the coefficients A_1, B_1, A_2, B_2 or of a_1, a_2, b_1, b_2.

At first, we can see that these coefficients are very small and only a series of very precise observations, over a long time period, is appropriate for their determination. This is fulfilled by the systematic latitude observations, especially those of the International Latitude Service (I.L.S.). Unfortunately, these observations were only published in 1934. It was impossible to use the whole set of observations, because in the exposition of changes in the program of some pairs of stars, less than one nutational period was observed. (By nutation we mean the change in obliquity of the ecliptic.) The observations that are available in general can be classified as follows: /98

A 26 pairs, observed during 1900-1934
B 27 pairs, observed during 1900-1922
C 21 pairs, observed during 1906-1934

In column 1 of Table 2 we give the numbers given to the pairs in the programs 1899-1905 and 1912-1922. For some of these pairs, the number was increased by one in the program 1922-1934 so that, for example, pair 65 was numbered 66 after 1922.7. These pairs are indicated by an asterisk.

The letters A, B, C denote the circles when the observations of the given pairs were done. The GC numbers of the stars compose the pairs given in column 3. Columns 4 and 5 give the declination and special motion of the center of the pair, computed from the information given in the catalogue. Pairs 42, 48, and 84 consist of stars not included in Boss' catalogue. Their declinations and special motion were obtained from the catalogues published by the I.L.S.

As is known, 96 pairs of the I.L.S. program (in which the observations were made before 1935) were separated into 12 groups with 8 pairs in each group. The closest values of the right ascensions of the centers of these groups are given in column 2 of Table 3. Columns 4 and 5 of Table 3 indicate the fractions of the year in which the mean values of the observations of the corresponding groups fall. There are two values, r_1, r_2, because since 1922.7, the observations have been obtained symmetrically with respect to midnight, which was not done before. /99
Consequently, the mean value of the observation for each group has changed.

We will in future denote by K the total number of years that have passed from 1900.0 until the beginning of the year of observation. Therefore the epoch (time) corresponding to the mean value of the observation of a given group can be expressed as follows:

$$1900.0 + K + r_1 \text{ before } 1922.7$$
$$1900.0 + K + r_2 \text{ after } 1922.7$$

TABLE 3.

Group	$\bar{\alpha}$	Pair	Mean Values of Observations τ_1	τ_2	Values of Ω in the Mean Value of the Time of Observation of the Group 1900	19227	$\Delta\tau\delta$
I	1^h	1-8	0.83	0.86	243°.0	179°.0	
II	3	9-16	0.93	0.75			
III	5	17-24	0.01	0.07			
IV	7	25-32	0.08	0.02			
V	9	33-40	0.15	0.10			
VI	11	41-48	0.22	0.18			
VII	13	49-56	0.29	0.26			
VIII	15	57-64	0.96	0.35			
IX	17	65-72	0.44	0.43			
X	19	72-80	0.52	0.51			
XI	21	81-88	0.62	0.60			
XII	23	89-96	0.73	0.68			

1. Pairs observed before 1922 and having the number 72 were /100 assigned the number 73. Since then, they changed from group IX to group X.

2. The negative value of r_2 for group III since 1923;the mean value of the observation of this group changed to 0.93 from the previous calendar year.

3. Δτδ is the reduction to the phenomenal position.

6.3.2. Reduction from the Original Data to a Common System of Declination and Special Motion (of Special Motions)

The original data collected for further computations was obtained from the following sources:

1. 1900-1905: Results of I.L.S. Volume 3. The declinations and special motions with which the latitude was computed were published in this volume. Original time 1903.0.

2. 1906-1908: Results of I.L.S. Volume 4. The corrections are in Volume 5.

3. 1909-1912: Results of I.L.S. Volume 5. The declinations and special motions are given for 1909.0.

4. 1912-1922: Results of I.L.S. Volume 6. The declinations and special motions are given in the same volume for the time 1915.0.

5. 1922.7-1939: Results of I.L.S. Volume 6. The declinations and special motions are given for 1928.0.

In this way, the instantaneous latitudes were published by the I.L.S. and were computed for the different circles with different original information given for declinations and special motions of the pairs. We take four systems for the following original epochs (times): 1903.0, 1909.0, 1915.0, and 1928.0. The mean date of observation of a pair in the different years is not exactly the same, but the limits for T_1 and T_2 remain /101 satisfactorily limited. Thus it is possible that these magnitudes do not exist separately for each year, but we can simply use their mean values computed for the first period (bdfore 1922.7) and the second (after 1922.7) relatively.

During the year, only three I.L.S. stations are considered, i.e Carloforte, Mizusawa, and Ukiah, that make observations without interruption. As original data collected, only the results of observations of these three stations were used.

First, for every year and for each pair, we form the mean value of the instantaneous latitudes. We obtained a long set (auay whole [sic]) of degrees, minutes and seconds, and we noted the rest in the original for the corresponding stations. Moreover, we obtained the value of the magnitude:

$$F_1 = 0{,}402C + 0{,}302M + 0{,}296u \qquad (2.11)$$

These are given in the sum. For the computation of these values, 135,000 instantaneous latitudes were used. The expression (2.11) looks like the usual expression of Kimura's term Z, which is used for the computation of the polar coordinates from the observations of three given stations. Hence it follows that the value F_1 does not depend on the motion of the pole.

It is necessary to note that in (2.11), C, M, U do not denote instantaneous latitudes, but the mean values obtained for each year separately for each pair. Thus in the change of F_1, the annual component will not be present (for the time being) because the mean value of the observation each year comes very close in the same part of the year. So only the slow latitude change remains nonperiodic, and the one of long period. For /102 the general analysis of the whole material, with the latitude change of long period as the object of our research, it was necessary that the results of different circles be reduced to a common system for the declinations and special motions. We obtained the system of Boss -- GC. The declinations of the 71 pairs were computed from the information given in the catalogue, and were found for the four original epochs (times) indicated in Table 4, and then the differences

$$\delta_{GC} - \delta_{LS}$$

where δ_{GC} is the declination according to GC and δ_{LS} that obtained directly from the I.L.S. publications. These differences are given in columns 2, 3, 4, and 5 of Table 4. The reductions of pairs 42, 48, 84 were extracted with original declinations and special motions obtained from the catalogue published in the "Results of the I.L.S." In Table 5, in columns 2, 3, 4, and 5, the differences in special motion and declination are given:

$$\mu_{GC} - \mu_{LS} = \Delta\mu.$$

The latitudes published in Volume 3 of the "Results of the I.L.S." were computed with the precession constant of Struve, and some corrections were required for them to be reduced to Newcomb's value. These corrections were extracted by the formula:

$$\Delta p_\delta = -0{,}00026\, p_\delta$$

where p_δ = annual precession in declination. These are given in Table 6 in multiples of 0".001.

The correction in the declination by which the value F_1 was reduced to a common system was computed by the formula: /103

$$\Delta\delta = (\delta_{GC} - \delta_{LS} + \Delta\mu z) + \Delta\mu (K - K_0)$$

where K_0 is the difference between the time of cataloguing and 1900.0. The terms inside the brackets keep a constant value within each circle. The last term was found simply by multiplication of $\Delta\mu$ by consecutive whole numbers.

For the first circle of observations, we use the formula

$$\Delta\mu + \Delta ps\,.$$

As we have shown before, the annual component of the term Z does not affect the change in magnitude F_1 if the mean time of the observation remains constant. During 1922.7, this time changed, and it was necessary that a very close calculation of the result F_1 be made. For the observations from 1922.7 until 1935.0, we use the formula for Z which was given by Kimura:

$$Z = 0''.019 \sin (2\theta - \alpha)\,.$$

where θ is the mean longitude of the Sun. Hence we can find the correction for the change in result of the Z term which we must put in all the latitudes obtained after 1922.7.

$$J = 0''.038 \sin (\theta_1 - \theta_2) \cos (\theta_1 + \theta_2 - \alpha)\,.$$

where θ_1 and θ_2 are the values of the Sun's longitude at times T_1 and T_2 after the beginning of the year. The correction J is expressed in 0".001 and is given in Table 6.

6.3.3. Corrections for Attraction Due to Jupiter and Saturn

The aberration due to a planet was computed by the following formula of Batterman:

$$\xi = c\,[\cos\lambda\,(\cos\vartheta . \sin\delta . \sin\alpha - \sin\vartheta . \cos\delta) - \sin\lambda . \sin\delta . \cos\alpha] \quad (2.15)$$

where θ is the declination of the ecliptic with respect to the equator (obliquity) α, δ are the right ascension and the declination of the star, λ is the longitude of the planet and c is a constant coefficient. If we suppose that the declination of the star is almost equal (very close) to the latitude, the formula can take the form: /104

$$\xi = -c\,[\sin\varphi . \sin(\lambda - \alpha) + \sin\vartheta . \cos\varphi . \cos\lambda] \qquad (2.16)$$

The coefficient c is 0".0086 for Jupiter and 0".0019 for Saturn. From formula (2.16) the corrections were computed from 1900 to 1922. For later years, the corrections were obtained from the tables that appear in the "Results of the I.L.S." Volume 8. Table 7 gives a summary of the values of ξ in 0".001.

6.3.4. Corrections for Small Nutational Terms /105

In 1912, Ross showed that in the calculations of the quantities A and B published in the Berliner Jahrbuch [Berlin Yearbook], not all small nutational terms had been taken into consideration as they should have been, for such precise calculations. He published a list of these terms and auxiliary tables for the calculation of the corrections, for the influence in the phenomenal right ascension and the phenomenal declination. Before 1922.7, Ross' corrections were introduced in the I.L.S. in the mean values of the groups, and after this year, in the separate latitude values. This introduced some errors that were not eliminated by later corrections. There was some confusion in the work of the I.L.S. which we will try to clear up. /106

For the first 6 years, the corrections for the influence of Ross' terms in the mean values of the group were computed by B. Wanach. He denoted the corrections by $\Delta\phi_o$, $\Delta\phi_1$, $\Delta\phi_2$ in which the index indicated the corresponding year (1900, 1901, etc.). The table for these corrections for the cycle of observations 1906.0 to 1911 appears in the "Results" Volume 5. These tables besides Ross' corrections give the value of the constant obtained, and the "Abberation" due to Jupiter and Saturn. These are denoted by $\Delta\phi_a$.

In the calculation of these corrections, Wanach, by mistake, took the same $\Delta\phi_a$ for all the years of observation, which is not correct. Mühlig observed that none of the I.L.S. publications mentioned this. Only the "Results of the I.L.S." mentioned that some errors existed in Wanach's tables, and that they must be replaced by Mühlig's tables. These tables are of the same form as those of Wanach; however, they satisfy the condition that the columns marked $\Delta\phi_o$, $\Delta\phi_1$, ..., contain Ross'

TABLE 7.

Year	Correction for group											
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
1900	+6	+5	0	-3	-5	-5	-4	-2	+1	+4	+6	+7
1	+5	+5	+2	-2	-5	-6	-7	-6	-4	-1	+2	+4
2	+3	+4	+3	-0	-3	-6	-8	-8	-7	-5	-2	+1
3	0	+2	+3	+2	-1	-4	-7	-9	-9	-8	-6	-3
4	-3	0	+3	+3	+1	-2	-5	-7	-9	-9	-8	-6
5	-5	-2	+2	+3	+3	-1	-1	-4	-6	-8	-8	-7
6	-5	-3	+1	+3	+4	+4	+2	-0	-3	-5	-6	-6
7	-4	-4	-1	+2	+4	+5	+5	+4	+2	-1	-3	-6
8	-2	-3	-2	0	+3	+5	+6	+6	+5	+3	+1	-1
9	-0	-1	-2	-1	+1	+3	+5	+6	+6	+6	+4	+2
10	+3	+1	-2	-2	-1	+1	+3	+5	+6	+6	+6	+4
11	+4	+2	-1	-2	-2	-2	0	+2	+4	+5	+6	+5
12	+4	+3	0	-2	-3	-4	-3	-2	0	+2	+3	+4
13	+3	+3	+1	-1	-3	-4	-5	-5	-4	-2	0	+1
14	0	+2	+2	+1	-1	-4	-6	-7	-7	-6	-4	-3
15	-2	0	+3	+2	0	-2	-4	-6	-8	-8	-6	-5
16	-5	-2	+2	+3	+2	+1	-1	-4	-6	-8	-8	-7
17	-6	-4	+1	+3	+4	+4	+2	0	-3	-6	-7	-7
18	-6	-5	-1	+2	+4	+6	+5	+4	+1	-2	-4	-6
19	-4	-4	-2	+1	+4	+6	+8	+7	+5	+3	0	-3
20	-2	-3	-3	-1	+2	+6	+8	+9	+9	+7	+4	+1
21	+2	-1	-4	-2	0	+3	+6	+9	+10	+9	+7	+5
22	+4	+1	-3	-3	-2	0	+4	+6	+9	+10	+9	+7
23	+6	+3	0	-3	-3	-2	0	+3	+6	+8	+8	+8
24	+6	+4	+1	-2	-4	-4	-3	-1	+2	+4	+6	+7
25	+5	+4	+2	-2	-4	-6	-6	-4	-2	0	+3	+5
26	+3	+3	+2	0	-3	-6	-7	-7	-6	-4	-1	-1
27	0	+2	+2	+2	-2	-4	-6	-7	-7	-6	-4	-2
28	-3	-1	+1	+2	+1	-1	-3	-5	-6	-7	-6	-5
29	-4	-2	0	+2	+2	+2	0	-2	-3	-5	-5	-5
30	-4	-3	-2	+2	+3	+4	+3	+2	0	-1	-3	-4
31	-2	-3	-2	0	+3	+4	+5	+5	+4	+2	0	-1
32	0	-1	-2	-2	+1	+3	+5	+6	+6	+5	+4	+2
33	+2	-0	-1	-2	0	+2	+3	+5	+6	+6	+6	+4
34	+4	+2	0	-2	-2	-1	+1	+3	+5	+6	+6	+6
35	+5	+3	+1	-2	-3	-3	-2	-1	+2	+4	+6	+6

corrections in addition to the ones due to aberration caused by Jupiter and Saturn. /107

Unfortunately, this does not exist anywhere. Even in the beginning of 1916, the formula used for computation of A, B (Berliner Jahrbuch) was not yet complete, and some small nutational terms, besides the numbers indicated by Ross, were added. Notice that neither Mankoff nor Kimura had observed this, and in the correction of the remarks, they continue to take into consideration all of Ross' terms as they appear in Volumes 6 and 7. This error was discovered only in 1952 by Uemae, as we have already mentioned, in Volume 4 of the "Results"; the corrections for the influence of the small terms, however, were not done for individual latitudes. From Uemae's work, we conclude that we can not use these values for corrections as given in this volume, for 1916 and the following years. These must be computed from the beginning. This is difficult for the information published in Volume 8 of the "Results." Ross' corrections were made here for all individual latitudes, but the computations were wrong. The omissions and the introduction of new corrections introduce many tedious calculations. Fortunately, we can avoid this, thanks to Uemae, who gave a table for the cycle 1922-34 for the difference in corrections $\Delta n = U - R$, where R is the value of the correction obtained if we take into consideration all Ross' terms, and U only those that had not been considered until 1916. The values Δn are given for mean values of the groups, and obtained separately for morning and evening observations. We have recomputed the corrections for the influence of the small nutational terms from 1900 to 1922. For this calculation, we used the tables of the /108
values of $\sin \varepsilon \delta_\lambda$ and δ_ε for the years 1900-1915, the ones published by Ross, and for the years 1916-22, those of Uemae.

The following are given as a check of the computations:

1. Wanach tables for 1900-1905.

2. Tables of the "Results" Volume 5 for 1906-1911.

3. The values $\Delta\phi$ in "Results" Volume 6, from which we excluded the correction for the "aberration" due to Jupiter and Saturn for the years 1912-1915.

The corrections for the years 1922-34 are obtained according to Uemae's work. In this work, for every time, the mean value of the corrections was formed for morning and night observations which Uemae gave separately. The results were summarized in Table 8, where n is expressed in 0".001.

Thus, we are able to obtain the corrected values of F by the formula:

$$F_2 = F_1 + D\delta + \xi + \eta + J$$

TABLE 8.

Year	Correction for group											
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
1900	+7	0	-7	-6	-4	-4	-6	-5	-2	+1	+4	+7
1	-1	-6	-4	-4	-2	0	-2	-4	-3	+1	+3	+2
2	-6	-12	-8	-6	-1	+4	+4	+4	+4	+6	+6	+2
3	-1	-8	-12	-9	-5	0	+4	+6	+6	+6	+6	+5
4	-3	-6	-8	-4	-2	+1	+2	+2	+2	+1	0	-2
5	-8	-9	-5	+1	+6	+10	+8	+5	+3	0	-4	-7
6	-6	-6	-6	0	+6	+11	+11	+8	+6	+3	0	-4
7	-2	-3	-4	0	+4	+6	+6	+5	+2	-1	-2	-1
8	-2	-2	+2	+7	+10	+8	+4	+2	-2	-7	-7	-2
9	0	0	+2	+9	+11	+8	+6	+2	-1	-4	-4	-1
10	+3	+3	+1	+4	+6	+6	+4	+1	0	-2	+1	+3
11	+2	+4	+6	+5	+4	+3	+2	-1	-6	-6	0	+2
12	+2	0	+6	+7	+5	+2	+2	-2	-6	-6	-2	+3
13	+4	+1	0	+2	+5	+2	+1	0	-2	-4	+2	+4
14	+5	+4	-1	0	+4	+1	0	-2	-4	-2	+2	+4
15	+2	0	+2	+1	0	+2	0	-5	-8	-4	0	+2
16	-2	-3	+4	+2	-1	+	+2	+2	+2	0	0	-1
17	+3	0	-2	-2	-2	-2	0	+2	+4	+2	+1	+2
18	+4	+4	-2	0	-2	-3	-4	-3	-2	-2	-1	+2
19	-2	-2	+2	+2	+2	+1	0	-2	-2	-2	-4	-2
20	-4	-5	-2	0	0	+2	+3	+4	0	+1	+1	0
21	+2	0	-4	-3	-2	-4	-2	+1	+4	0	+2	+4
22	-2	+2	+8	0	-1	-4	-4	-4	-4	-4	-2	-1
23	0	+4	+8	+8	+4	-2	-5	-6	-5	-4	-2	-1
24	+2	+4	+7	+6	0	-4	-7	-7	-5	-2	0	+2
25	+2	+4	+6	+5	-2	-6	-8	-7	-4	0	+2	+2
26	+2	+3	+4	+3	-2	-6	-8	-6	-2	+2	+4	+3
27	+2	+2	+1	0	-2	-6	-6	-4	-0	+4	+4	+2
28	0	0	0	-1	-4	-6	-6	-3	+2	+6	+4	0
29	0	0	-2	-2	-4	-6	-5	-2	+3	+6	+4	+2
30	-2	-2	-2	-2	-3	-4	-4	0	+4	+6	+4	0
31	-4	-3	-3	-2	-4	-4	-3	0	+6	+6	+2	-2
32	-3	-4	-2	-3	-2	-3	-2	+2	+6	+6	+2	-2
33	-4	-4	-3	-2	-2	-2	-2	+2	+6	+5	0	-3
34	-6	-5	-2	-1	0	-2	-1	+3	+6	+3	0	-3

6.3.5. Calculation of the Zenith Distances of the Center of the Pair

One of the most demanding phases of the computations was the determination of the corrections for the value of the (eyepiece) micrometric knob. For this determination, it is necessary that we have the zenith distance of a pair. The zenith distance is expressed in micrometric turns; it can be written as follows:

/110

$$m = \frac{\varphi - \delta}{R} = \frac{1}{R}\left[\varphi - \delta_{1900,0} - AV\,(k+\tau) - \Delta_\tau \delta\right] \qquad (2.18)$$

where ϕ is the latitude, $\delta_{1900.0}$ is the mean declination of the center of the pair at the beginning of 1900. AV is the annual change of the mean declination, and $\Delta\tau\delta$ is the reduction for the phenomenal position.

Because the value R for all the stations (3) can be considered that was taken to be the same and equal to 39".74, we will have:

$$\frac{1}{R} = 0.02516$$

Moreover, the instantaneous value of the latitude was given as the fixed value 39°8'8" in the calculations $\Delta\tau$ was taken into consideration only as a fact of aberration, in which case

$$\Delta_\tau\delta = -20''.47\left[\cos\Theta \sin\varphi + (\sin\Theta \cos\alpha - \cos\Theta . \cos\vartheta . \sin\alpha)\sin\varphi\right]$$

We introduce the following note:

$$m_0 = \frac{1}{R}(\varphi - \delta_{1900.0}) \quad \frac{dm}{dt} = \frac{AV}{R}, \quad \alpha = \frac{\Delta_\tau\delta}{R}, \quad m_0' = m_0 - \frac{dm}{dt}\tau - \alpha.$$

and formula (2.18) becomes

$$m = m_0' - \frac{dm}{dt}\kappa$$

The values $\Delta\tau\delta$ are given in groups in column 8 of Table 8. Table 9 gives the m values for each pair for the beginning and end of the cycle of observation.

6.3.6. Determination of the Correction for the Mean Value of the Micrometric Knob, or the Mean Graduated Value

It was necessary to determine the errors in the following magnitude, which we call mean value of the micrometric knob, or mean graduated value:

$$R = 0{,}402\, R_c + 0{,}302\, R_M + 0{,}296\, R_U$$

where R_C, R_M, R_U are the graduated values of the three stations.

TABLE 9. /111

Pair	m 1900	m 1906	m 1922	m 1934	$\frac{dm}{dt}$
1	—	+ 8.8	—	− 5.3	− 0.504
2	+ 4.7	. . .	− 5.8	—	− 0.501
3	+ 5.3	. . .	− 5.1	—	− 0.497
5	+ 2.2	. . .	− 7.9	—	− 0.479
8	+ 1.2	. . .	− 8.0	—	− 0.440
9	− 3.1	. . .	− 12.0	—	− 0.420
10	—	+ 7.4	. . .	− 3.9	− 0.403
11	+ 9.9	. . .	. . .	− 2.5	− 0.378
13	+ 11.5	. . .	. . .	+ 0.3	− 0.340
14	− 2.5	. .	− 9.1	—	− 0.317
15	− 6.2	. . .	− 12.3	—	− 0.292
16	—	+ 1.3	. . .	− 6.2	− 0.268
17	+ 11.8	. . .	. . .	+ 3.9	− 0.233
18	+ 1.5	. . .	. . .	− 5.7	− 0.212
19	—	+ 4.7	. . .	− 0.2	− 0.177
20	− 7.0	. . .	− 10.2	—	− 0.149
21	− 0.1	. . .	. . .	− 4.0	− 1.115
22	− 0.6	. . .	. . .	− 3.8	− 0.095
23	− 5.0	. . .	. . .	− 7.1	− 0.062
24	—	+ 3.5	. . .	+ 2.8	− 0.024
25	+ 6.0	. . .	. . .	+ 6.9	+ 0.021
26	+ 4.8	. . .	. . .	+ 6.9	+ 0.061
27	—	− 8.0	. . .	+ 5.2	+ 0.095
28	− 1.2	. . .	. . .	+ 3.1	+ 0.126
29	—	+ 2.4	. . .	+ 6.9	+ 0.160
32	− 7.8	. . .	. . .	+ 0.7	+ 0.251
34	—	+ 5.2	. . .	+ 3.3	+ 0.304
35	− 10.1	. .	− 2.5	—	+ 0.346
36	+ 0.4	. . . .	+ 8.4	—	+ 0.368

(continued)

37	− 1.2	. . .	+ 7.2	—	+ 0.379
38	—	− 2.8	. . .	+ 8.2	+ 0.394
39	+ 2.4	. . .	+ 11.6	—	+ 0.417
40	− 4.6	. . .	+ 5.1	—	+ 0.444
41	− 0.5	. . .	+ 10.2	—	+ 0.458
42	—	− 2.4	. . .	+ 11.6	+ 0.469
44	− 11.5	. . .	. . .	+ 5.0	+ 0.487
47	+ 1.5	. . .	+ 12.6	—	+ 0.503
48	—	− 9.8	. . .	+ 4.4	+ 0.505
50	− 5.7	. . .	+ 5.2	—	+ 0.495
51	− 4.2	. . .	+ 6.6	—	+ 0.494
52	—	− 11.6	. . .	+ 2.1	+ 0.488
56	—	− 5.1	. . .	+ 7.5	+ 0.440
57	− 0.7	. . .	+ 8.5	—	+ 0.417
58	− 11.9	. . .	. . .	+ 1.7	+ 0.402
61	+ 2.4	. . .	+ 9.3	—	+ 0.314
62	3.0	. . .	+ 3.4	—	+ 0.290

/112

Pair	m				dm/dt
	1900	1906	1922	1934	
64	+ 4.3	. . .	+ 9.6	—	+ 0.240
65	—	+ 2.3	. . .	+ 8.3	+ 0.214
67	+ 2.3	. . .	. . .	+ 6.9	+ 0.134
68	—	− 0.5	. . .	+ 2.5	+ 0.107
69	—	− 3.3	. . .	− 1.2	+ 0.075
70	− 6.1	. . .	. . .	− 4.7	+ 0.041
71	− 2.4	. . .	. . .	− 1.9	+ 0.017
72	− 5.0	. . .	. . .	− 5.5	− 0.014
73	+ 0.7	. . .	. . .	− 0.5	− 0.034
74	− 3.2	. . .	. . .	− 5.7	− 0.073
75	—	+ 5.5	. . .	+ 2.7	− 0.101
76	+ 2.2	. . .	. . .	− 3.1	− 0.156
77	+ 3.6	. . .	. . .	− 2.5	− 0.179
78	+ 4.1	. . .	. . .	− 3.0	− 0.209
80	− 3.5	. . .	− 9.5	—	− 0.271
82	+ 5.5	. . .	. . .	− 5.0	− 0.309
83	− 0.3	. . .	− 7.7	—	− 0.339
84	—	+ 5.8	. . .	− 4.1	− 0.355
85	+ 4.9	. . .	. . .	− 7.8	− 0.375
86	—	+ 5.5	. . .	− 5.5	− 0.397
87	+ 2.1	. . .	− 7.0	—	− 0.414
88	− 2.1	. . .	− 11.5	—	− 0.427
89	− 0.9	. . .	− 10.6	—	− 0.442
90	—	+ 4.7	. . .	− 8.1	− 0.458
91	—	+ 5.6	. . .	− 7.6	− 0.471
92	+ 3.8	. . .	− 5.8	—	− 0.434
96	+ 7.1	. . .	. . .	− 10.1	− 0.504

We found earlier that the R values are used in the work of the I.L.S. and in some cases they are obviously wrong. It is not possible to find their exact values, but we can use some information given about their general behavior. This is necessary because some errors in the values are obtained for R, and they completely change the curve of the nonpolar variable of latitude, taken in the observations of the separate pairs, and they may very significantly affect the final results. /113

After trying various methods, we define in terms of the following known facts a basis for the comparison of the mean latitudes obtained from observations of the pairs with zenith distances of opposite points.

We choose some pairs with large positive values of zenith distance Z. The mean value of these Z values for these pairs is expressed in turnings of the micrometer knob. We denote this by m_+ and the mean value of the F_2 values by S_+. We denote by m_- and S_- the corresponding quantities with negative zenith distances. Then the correction in R can be determined by the formula

$$\Delta R = \frac{S_- - S_+}{m_+ - m_-} \tag{2.22}$$

This will give the exact mean graduated value only if the declinations are absolutely exact. Of course this does not happen. Moreover, in the exposition of the errors of special motion ΔR, they were determined in such a way that they can have a false linear behavior. In the first stage of the calculation, we can afford these errors, because we are only interested in the periodic intervals of the change F_2. It was important to notice that the corrections were completely independent of the changes in latitude of a nutational character. On the other hand, it could happen that after the introduction of the changes, these changes were negligible. Usually this can not happen in the collection of the pairs; for the determination of ΔR we can observe the following principle: /114

In each of the two groups of pairs (so that we have positive and negative zenith distances), the mean values of sinα and cosα must tend to zero. First we compute ΔR separately for the cycles 1900-22 and 1906-34. Table 10 includes some data given for the pairs of groups used.

The catalogue of pairs composes the groups, as in Table 11. The original data for the calculation of the ΔR corrections and of these corrections are expressed in 0".001, and are given in Table 12. Because the determination was done separately for

TABLE 10.

Beginning of cycle of observations End of cycle of observations.	1900 1922		1906 1934	
Number of pairs in the group	12	15	15	9
Mean of zenith distances of pairs in the group at the beginning of the cycle in rotations of the micrometer screw	+4,92	−5,22	+3,78	−4,74
The same for the end of the cycle	+5,43	−5,29	+3,30	−4.12
Mean value of cos α	−0,04	+0,01	+0,01	−0.04
-//- -//- -//- sin α	−0,03	+0,03	−0,03	−0.04
-//- -//- -//- T	0.28	0.24	0.27	0.25

TABLE 11.

1900–1922				1906–1934			
Pair	m_o	Pair	m_o	Pair	m_o	Pair	m_o
13	+11,5	9	− 3.1	11	+ 9.9	18	+ 1.5
17	+11,3	20	− 7.0	13	+11.5	22	− 0.6
25	+ 6,0	23	− 5.0	17	+ 11.8	23	− 5.0
26	+ 4.8	30	− 5.1	25	+ 6.0	27	− 8.6
36	+ 04	32	− 7.8	26	+ 4.8	52	− 14.5
39	+ 24	35	−10.1	29	+ 14	70	− 6.1
61	+ 2.4	44	−11.5	38	− 58	71	− 2.4
64	+ 43	57	− 0.7	42	− 5.2	72	− 5.0
67	+ 2,3	58	−11.9	65	+ 1.0	74	− 3.2
77	+ 3.5	70	− 6.1	67	+ 2.3		
78	+ 4.1	74	− 32	68	− 1.2		
82	+ 5.5	80	− 3.5	75	+ 6.1		
		83	− 0.3	77	+ 3.6		
		88	− 2.1	78	+ 4.1		
		89	− 0.9	84	+ 8.0		

TABLE 12. /116

Year	1900 – 1922 S₋	1900 – 1922 S₊	1900 – 1922 ΔR₁	1906 – 1934 S₋	1906 – 1934 S₊	1906 – 1934 ΔR₂	ΔR₁-ΔR₂	ΔR
1900								– 39
1	929	1001	– 71	—	—	—	—	– 66
2	870	970	– 98	—	—	—	—	– 22
3	899	954	– 54	—	—	—	—	– 20
4	922	975	– 52	—	—	—	—	– 3
5	917	953	– 35	—	—	—	—	– 4
6	897	934	– 36	—	—	—	—	+ 16
7	891	913	– 21	888	869	+ 22	– 43	– 16
8	868	915	– 45	851	867	– 19	– 26	– 51
9	846	928	– 79	838	884	– 55	– 24	– 72
10	862	979	– 112	853	906	– 63	– 49	– 24
11	883	944	– 58	868	882	– 22	– 36	– 74
12	848	963	– 110	827	884	– 69	– 41	– 150
13	799	992	– 184	782	904	– 148	– 36	– 154
14	817	1009	– 183	800	929	– 157	– 26	– 206
15	795	1033	– 226	765	943	– 218	– 8	– 186
16	801	1042	– 228	783	927	– 177	– 51	– 246
17	782	1076	– 278	784	983	– 246	– 32	– 210
18	797	1040	– 229	776	955	– 222	– 7	– 215
19	773	1034	– 246	759	932	– 216	– 30	– 218
20	751	1026	– 258	744	911	– 210	– 48	– 210
21	739	995	– 239	731	901	– 214	– 25	– 207
22	—	—	—	732	895	– 207	—	– 11
23	—	—	—	839	484	– 11	—	+ 1
24	—	—	—	852	851	+ 1	—	– 45
25	—	—	—	827	862	– 45	—	– 28
26	—	—	—	839	861	– 23	—	—
27	—	—	—	—	—	—	—	+ 28
28	—	—	—	867	846	+ 28	—	– 18
29	—	—	—	864	878	– 18	—	– 24
30	—	—	—	850	868	– 24	—	+ 4
31	—	—	—	863	860	+ 4	—	– 16
32	—	—	—	848	860	– 16	—	– 17
33	—	—	—	851	864	– 17	—	+ 12
34	—	—	—	869	860	+ 12	—	

the two series of observations, we obtain two rows of values, i.e. $\Delta R_1'$ (cycle 1900-22) and ΔR_2 (cycle 1906-34). It is obvious that ΔR_1 and ΔR_2 present systematic differences. /115

In order to combine the two series, we find the mean differences:

$$\Delta R_1 - \Delta R_2 = -0''.0032.$$

The direction in the change $\Delta R_1' - \Delta R_2$ does not show. So we can simply add to all ΔR values a constant 0".0032, and obtain as a final correction the graduation of values:

$$\begin{array}{ll} 1901-05 & \Delta R_1 + 0''.0032 \\ 1907-21 & \frac{1}{2}(\Delta R_1 + \Delta R_2) + 0''.0016 \\ 1922-34 & \Delta R_2 \end{array}$$

These ΔR values are given in Table 12. From these, we obtain Graph 3. In this graph, the jump of ΔR in 1922 is clearly shown. The general path of the values between 1905-1915 shows that a discontinuous change probably takes place in 1922. In order to see if this is actually so, we follow the graph more closely for 1909-1911, taking the following considerations into account. The ΔR values are computed from (2.22) and denote corrections in R, taken on the average for the following time series of observations. /117

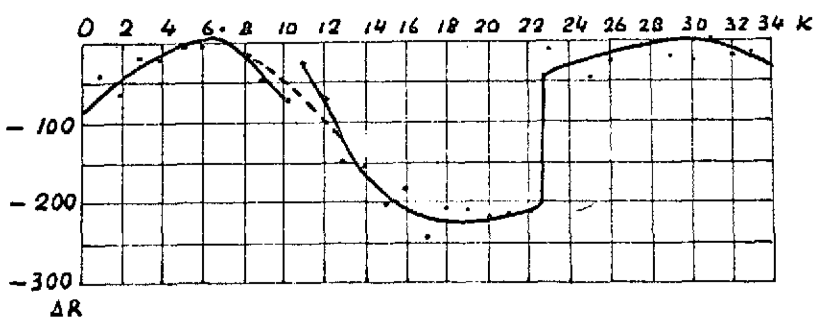


Figure 3.

When the sequence of values of ΔR makes it possible for us to construct a smooth curve, then we have no difficulty with any pair. Difficulty can be expected from the points at which the graph breaks (discontinuities). In order to go into more detail for 1909-1911, we find the values S_+ and S_- in this interval for consecutive time intervals, each of less than a year. The results are given in Table 13.

TABLE 13.

1900 - 1922				1906-1934			
t_m	S	t_m	S_+	t_m	S_-	t_m	S_+
1910.24	862	1910.28	979	1910.25	853	1910.27	906
.31	859	.36	981	.36	849	.33	904
.38	858	.44	977	.47	844	.40	907
.44	859	.53	973	.58	844	.47	905
.51	859	.61	967	.69	850	.53	901
.58	863	.69	967	.81	856	.60	897
.64	869	.78	963	.92	866	.67	898
.71	875	.86	961	1911.03	868	.73	903
.78	873	.94	955	.14	873	.80	905
.84	879	1911.03	956	.25	868	.87	903
.91	885	.11	952	.36	873	.93	903
.98	885	.19	944	.47	875	1911.00	903
1911.04	882	.28	944	.58	875	.07	898
.11	883	.36	951	.69	863	.13	895
.18	883	.44	952	.81	851	.20	888
.31	884	.53	953	.92	841	.27	882
.38	884	.61	948	1912.03	836	.33	886
.44	887	.69	950	.14	831	.40	887
.51	887	.78	956	.25	827	.47	888
.58	884	.86	952	—	—	.53	884
.64	884	.94	951	—	—	.60	879
.71	880	1912.03	955	—	—	.67	874
.78	880	.11	961	—	—	.73	870
.84	880	.19	960	—	—	.80	870
.91	868	.28	963	—	—	.93	872
.98	863	—	—	—	—	1912.00	875
1912.04	861	—	—	—	—	.07	880
.11	852	—	—	—	—	.13	880
.18	850	—	—	—	—	.20	883
.24	848	—	—	—	—		

By using the data of this table, we construct smoothed graphs of the change of S_+ and S_-, and from these we obtain, for every ten intervals of the year, the differences $S_- - S_+$, and so we can use (2.22) again; but now the results of the computation of the formula are denoted by I instead of ΔR. The reason for this change is given below. /118

TABLE 14. /119

Epoch	I	$\frac{dI}{dt}$	ΔR	Epoch	I	$\frac{dI}{dt}$	ΔR
1909.8	-75			1911.4	-38	-110	+24
.9	-78	+10	-65	.5	-50	-110	+19
1910.0	-75	+40	-100	.6	-60	-100	+4
.1	-69	+80	-150	.7	-70	—	-44
.2	-60	+80	-116	.8	—	—	-50
.3	-52	+80	-81	.9	—	—	-55
.4	-44	+70	-46	1912.0	—	—	-60
.5	-38	+60	-41	.1	—	—	-70
.6	-31	+60	-56	.2	—	—	-76
.7	-26	+40	-84	.3	—	—	-81
.8	-22	+20	-70	.4	—	—	-86
.9	-22	0	-55	.5	—	—	-91
1911.0	-22	0	-60	.6	—	—	-96
.1	-21	0	-70	.7	—	—	-102
.2	-23	-40	-36	.8	—	—	-107
.3	-28	-80	-1	.9	—	—	-112

If we express the time t in years, the mean value in 1 year of the correction of R is obtained from (2.22) and can be given approximately as:

$$I = \int_t^{t+1} \Delta R_t \, dt$$

In the case of linear change, $I = \Delta R_t + 0.5$. Of course, for the time intervals where we consider it as a linear function, the approximation of ΔR is not exact. Then, for the computation of ΔR_+ we can use the approximation

$$\Delta R_t = \Delta R_{t+1} - \frac{dI}{dt} \Big/ \qquad (2.24)$$

but then it is necessary that we know ΔR_{t+1}. The later conditions are fulfilled if we approach gradually the part of the curve we are studying from a section in the neighborhood for which the change in R can correctly be represented by a linear function of time.

The part of the graph for 1911-12 has this property, and we start our computations from there, going backward along the x-axis. The results of the computation are given in Table 14. In the last column of this table, the values of ΔR, starting from 1911.7 are obtained by the construction of an earlier graph, /120 but the rest are obtained by an extension according to (2.24). We will give a result. In order to obtain the value of ΔR for 1911.3, we first take from Table 14 the value for 1912.3. This is (-0".0081). Then, corresponding to the date 1911.3, we find dI/dt = -0".0080, from which we obtain the following quantity:

$$\Delta R_{1911.3} = -0''.0081 + 0''.0080 = -0''.0001.$$

From the data of Tables 12 and 14, we form a large gradual graph of ΔR and from this we obtain the final corrections, which are given in Table 15 (in 0".001). The fact that the method can not be rigorously expressed in words gives the corrections, but their chances alone do not affect the final results very much. But this is not certain a priori, because it concerns the false linear changes of ΔR, which can form a sequence of errors in the special motions of the pairs of stars used for the determination of the corrections. (These results can be studied in detail below.)

TABLE 15.

R	Corrections for group											
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
0	−6	−6	−7	−7	−7	−7	−7	−7	−6	−6	−6	−6
1	−5	−5	−6	−6	−6	−6	−6	−6	−5	−5	−5	−5
2	−4	−4	−5	−5	−4	−4	−4	−4	−4	−4	−4	−4
3	−2	−2	−3	−3	−3	−3	−3	−3	−3	−3	−2	−2
4	−1	−1	−2	−2	−2	−2	−2	−2	−2	−1	−1	−1
5	0	0	−1	−1	−1	−1	−1	−1	0	0	0	0
6	+1	+1	0	0	0	0	0	0	0	0	0	+1
7	0	−1	+1	+1	0	0	0	0	0	0	0	0

(cont'd.)

Table 15, cont'd.

8	−4	−4	−1	−1	−1	−2	−2	−2	−2	−3	−3	−3
9	−9	−10	−4	−4	−5	−5	−5	−6	−6	−6	−7	−8
10	−6	−6	−10	−14	−14	−11	−8	−6	−5	−4	−5	−6
11	−5	−6	−6	−6	−5	−3	0	+2	+2	+1	−1	−3
12	−10	−11	−6	−6	−7	−7	−8	−8	−9	−9	−9	−10
13	−15	−16	−11	−12	−12	−12	−13	−13	−14	−14	−14	−15
14	−18	−19	−16	−16	−16	−17	−17	−17	−17	−18	−18	−18
15	−20	−20	−19	−19	−19	−19	−19	−19	−20	−20	−20	−20

R	Corrections for group											
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
16	−21	−21	−20	−20	−20	−20	−20	−21	−21	−21	−21	−21
17	−22	−22	−21	−21	−21	−21	−21	−21	−21	−21	−21	−22
18	−22	−22	−22	−22	−22	−22	−22	−22	−22	−22	−22	−22
19	−22	−22	−22	−22	−22	−22	−22	−22	−22	−22	−22	−22
20	−21	−21	−22	−22	−22	−22	−21	−21	−21	−21	−21	−21
21	−21	−21	−21	−21	−21	−21	−21	−21	−21	−21	−21	−21
22	−3	−3	−21	−21	−21	−21	−21	−21	−21	−21	−21	−12
23	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
24	−2	−2	−3	−3	−3	−3	−3	−3	−3	−3	−2	−2
25	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
26	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
27	−1	−1	−2	−2	−2	−2	−2	−2	−2	−1	−1	−1
28	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
29	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
30	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
31	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
32	0	0	−1	−1	−1	−1	−1	−1	−1	−1	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0

/121

The method described does not allow a detailed study of the change in ΔR. So it is probably necessary to reconsider the whole body of I.L.S. material, and complicated computations would be necessary. In general, we think that this would be very important if it were done, because the errors were discovered by us (Fedorov), showing that the method used by I.L.S. does not allow the determination of the gradual values with satisfactory accuracy. For our immediate purpose, however, such a detailed analysis is not important. We could limit ourselves to the examination of the general path of ΔR. The numbers giving the values of F_2 corrected for errors in the mean scale of values are obtained by the formula

$$F_3 = F_2 + m\Delta R$$

and are given in a table (see Fedorov).

6.3.7. Nonperiodic Changes in Latitude /122

In order to diminish arbitrary errors in the following harmonic analysis of F_3, we must determine and exclude the nonperiodic changes of this magnitude. With this, we compose the mean annual values of F_3 separately for the pairs in the groups A, B, and C. Their differences from 0".900 are denoted by S_A, S_B, S_C, and are given in columns 2-4 of Table 16 (0".001). The systematic differences in these quantities can be given as:

$$S_A - S_B = \Delta_{AB} = 0.''0048 - 0.''0009\,k$$

$$S_A - S_C = \Delta_{AC} = -0.''0051 - 0.''0011\,k$$

Then we reduce the results obtained for the other two groups to those obtained from group A. Thus we get the quantities:

$$S'_B = S_B + \Delta_{AB} \quad , \quad S'_C = S_C + \Delta_{AC}$$

which are given in Table 16.

As final values of the corrections for the "slow" changes of F_3 common to all pairs, we obtain the following mean values: /123

$$\Delta S = -\frac{S_A + S'_B}{2}\ (1900\text{-}05) \qquad \Delta S = -\frac{S_A + S'_B + S'_C}{3}\ (05\text{-}21) \qquad \Delta S = -\frac{S_A + S'_C}{2}\ (22\text{-}34)$$

TABLE 16.

R	S_A	S_B	S_C	S'_B	S'_C	ΔS
0	+ 28	+ 43	—	+ 39	—	+ 34
1	+ 50	+ 53	—	+ 50	—	+ 50
2	+ 7	+ 10	—	+ 8	—	+ 8
3	+ 16	+ 21	—	+ 20	—	+ 18
4	+ 41	+ 29	—	+ 29	—	+ 35
5	+ 20	+ 27	—	+ 28	—	+ 24
6	+ 16	− 1	+ 14	+ 1	+ 13	+ 10
7	0	− 7	− 3	− 4	− 5	− 3
8	− 9	− 26	− 4	− 22	− 7	− 13
9	− 24	− 1	− 6	+ 4	− 10	− 10
10	+ 5	+ 6	+ 16	+ 13	+ 12	+ 10
11	− 5	− 7	+ 2	+ 1	− 3	− 2
12	− 3	− 29	+ 1	− 20	− 5	− 9
13	− 18	− 21	− 14	− 11	− 21	− 17
14	+ 4	− 6	+ 14	+ 5	+ 6	+ 5
15	− 6	− 4	+ 8	+ 8	− 1	0
16	+ 9	− 20	+ 12	− 7	+ 2	+ 1
17	+ 17	+ 14	+ 29	+ 28	+ 18	+ 21
R	S_A	S_B	S_C	S'_B	S'_C	ΔS
18	− 8	− 18	+ 7	− 3	− 5	− 5
19	− 10	− 42	− 4	− 26	− 17	− 18
20	− 45	− 47	− 40	− 30	− 54	− 43
21	− 57	− 78	− 43	− 60	− 58	− 58
22	− 57	—	− 45	—	− 61	− 59
23	− 39	—	− 20	—	− 37	− 38
24	− 35	—	− 17	—	− 35	− 35
25	− 42	—	− 19	—	− 37	− 40
26	− 40	—	− 9	—	− 28	− 34
27	− 34	—	− 16	—	− 36	− 35
28	− 24	—	− 4	—	− 25	− 24
29	− 28	—	− 4	—	− 26	− 27
30	− 37	—	− 11	—	− 34	− 36
31	− 35	—	− 15	—	− 39	− 37
32	− 47	—	− 19	—	− 34	− 40
33	− 44	—	− 16	—	− 42	− 43
34	− 37	—	− 18	—	− 45	− 41

/123

These are shown in a graph. The values Δs are obtained from a smooth curve and are given in Table 17.

The values $F_4 = F_3 + \Delta s$ are given in the table (see Fedorov).

TABLE 17.

K	Corrections for group											
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
0	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30
1	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30
2	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30
3	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30	−30
4	−28	−27	−30	−30	−30	−29	−29	−29	−29	−28	−28	−28
5	−19	−18	−27	−26	−25	−25	−24	−23	−22	−22	−21	−20
6	−5	−4	−17	−16	−15	−14	−12	−11	−10	−9	−8	−6
7	+4	+4	−3	−2	−2	−1	0	0	+1	+2	+2	+3
8	+8	+8	+5	+5	+6	+6	+6	+6	+7	+7	+7	+7
9	+8	+8	+8	+8	+8	+8	+8	+8	+8	+8	+8	+8
10	+8	+8	+8	+8	+8	+8	+8	+8	+8	+8	+8	+8
11	+8	+8	+8	+8	+8	+8	+8	+8	+8	+8	+8	+8
12	+7	+7	+8	+8	+8	+8	+8	+8	+8	+7	+7	+7
13	+5	+4	+7	+7	+7	+6	+6	+6	+6	+5	+5	+5
14	+1	0	+4	+4	+3	+3	+3	+2	+2	+2	+1	+1
15	−4	−5	0	0	−1	−1	−2	−2	−3	−3	−3	−4
16	−7	−8	−5	−5	−5	−6	−6	−6	−6	−7	−7	−7
17	−5	−4	−8	−8	−7	−7	−7	−6	−6	−6	−5	−5
18	+10	+11	−4	−3	−1	0	+2	+3	+4	+6	+7	+9
19	+27	+28	+13	+14	+16	+17	+19	+20	+21	+23	+25	+26
20	+46	+48	+30	+32	+33	+35	+37	+38	+40	+41	+43	+45
21	+57	+58	+50	+51	+52	+52	+53	+54	+54	+55	+56	+57
22	+50	+49	+59	+58	+57	+56	+55	+54	+54	+53	+52	+51
23	+40	+39	+48	+47	+46	+45	+44	+43	+43	+42	+42	+41
24	+36	+36	+38	+38	+38	+38	+37	+37	+37	+37	+36	+36
25	+36	+36	+36	+36	+36	+36	+36	+36	+36	+36	+36	+36
26	+35	+35	+36	+36	+36	+36	+36	+36	+35	+35	+35	+35
27	+32	+32	+35	+35	+34	+34	+34	+33	+33	+33	+33	+32
28	+30	+30	+32	+32	+32	+32	+32	+31	+31	+31	+31	+30

/124

Table 17, cont'd.

29	+32	+32	+30	+30	+31	+31	+31	+31	+31	+32	+32	+32
30	+34	+34	+32	+32	+32	+33	+33	+33	+34	+34	+34	+34
31	+38	+39	+35	+35	+35	+36	+36	+36	+37	+37	+37	+38
32	+42	+42	+39	+39	+40	+40	+40	+40	+41	+41	+41	+41
33	+43	+43	+42	+42	+42	+42	+42	+42	+43	+43	+43	+43
34	+43	+43	+43	+43	+43	+43	+43	+43	+43	+43	+43	+43

6.3.8. Determination of the Corrections for the Declination and Special Motion (A First Approach)

We can now consider that the initial data are ready for the determination of the coefficients of the nutational terms. If we put

$$F_4 = x + ky + \nu \tag{2.25}$$

where ν denotes the right-hand side of (2.6) or (2.8), the problem reduces to that of the determination of six unknowns. From these, the constant x can be considered as the error in the declination for the epoch 1900.0, and the coefficient y as the error of the special motion of a pair in the declination. These errors may, of course, differ in different pairs.

We can attempt the solution in two ways. In the first, we would be able to determine from the observations of each pair separately all the unknowns (six) and after the composition of the /125
values of the coefficients from the periodic terms in (2.6) or (2.8), to find the most probable values. For this, it is necessary to solve generally 74 systems of equations with six unknowns. This method was used by Przybyllok, but his case was different from ours, because he had to find not six but three unknowns, i.e. he had the determination of N, n and the phase difference (delay) which was considered as given.

In the second method, the problem can be solved by the following method of consecutive approximations. First, we find for each pair separately the x and y, and we neglect the periodic part (term) ν in (2.25). When this is done, we determine the coefficients of the periodic terms in (2.6) or (2.8) by using the rest of the deviations ν, not for separate pairs but the same for all the observations. Later if it is necessary, we can exclude from F_4 the periodic terms, and repeat the same procedure for a second approximation. We choose this method for its relative simplicity.

So for the determination of the fixed correction x, and the correction for the special motion of each pair, we have the system of equations

$$x_1 + K\psi_1 = F_4$$

where K = ℓ, ℓ + 1, ℓ + 2, ... m. The physical equations can be written in the form:

$$(m - \ell + 1)x_1 + \sum_{\ell}^{m} K y_1 = \sum_{\ell}^{m} F_4, \quad \sum_{\ell}^{m} K x_1 + \sum_{\ell}^{m} K^2 \psi_1 = \sum_{\ell}^{m} K F_4$$

and their solutions can be written as follows:

$$x_1 = p \sum_{\ell}^{m} F_4 + q \sum_{\ell}^{m} K F_4, \quad y_1 = p_1 \sum_{\ell}^{m} K F_4 + q \sum_{\ell}^{m} F_4 \tag{2.20}$$

where the coefficients p, p_1, q are given in Table 18. The values of the correction ($-x_1$) are given in Table 4 (column 6) and those of ($-y_1$) in Table 5 (column 6). The differences $F_4 - (x_1 + ky_1)$ are denoted by F_5 and are given in the table (see Fedorov). /126

TABLE 18.

R	$P \times 10^2$	$q \times 10^2$	$P_1 \times 10^4$
0-21	+16.996	-1.185	+11.293
0-22	+16.304	-1.086	+9.881
0-34	+10.952	-0.476	+2.801
6-34	+23.152	-0.985	+4.926

6.3.9. Determination of the Coefficients a_1, a_2, b_1, b_2

The values found for F_s were used for the determination of the nutational terms. We write this in the form

$$F_s = a_1 \cos(\Omega - \alpha) + b_1 \sin(\Omega - \alpha) + a_2 \cos(\Omega + \alpha) + b_2 \sin(\Omega + \alpha) \quad (2.27)$$

First we distribute all the F_s values by the phase Ω - a and then by the phase Ω + a. These phases are expressed in hours and are given in Table 30. In order to simplify the computations, we proceed slightly differently. All the pairs formed eight groups in such a way that the mean right ascensions of the centers of the groups were 0, 3, 6, 9, 12, 13, 18, 21 hours. After that, we obtained the mean value of F_s for the pairs of each group. These we call M_1 (for the 1900-1931 cycle) and M_2 (for the 1906-1934 cycle). These are given in Table 19. These values are distributed according to the phases Ω - a and Ω + a, as shown in the table, and for each phase the mean value is formed again. The results are given in Table 20. By haromonic analysis of the results of Table 20, we obtain the following values of the coefficients. /127

1900-21

$$a_1 = -0''0027 \pm 0''0017 \qquad b_1 = -0''0108 \pm 0''0017$$

$$a_2 = -0''0048 \pm 0''0021 \qquad b_2 = +0{,}0031 \pm 0{,}0021$$

1906-1934

$$a_1 = -0''0034 \pm 0{,}0013 \qquad b_1 = -0''0124 \pm 0''0013$$

$$a_2 = -0{,}0051 \pm 0{,}0016 \qquad b_2 = -0''0015 \quad 0''016$$

The results of two series of observations agree satisfactorily. It is true that these results are not totally independent of each other, but the observed data common to both series form only one quarter of the entire original material. Giving the mean value of the results obtained from the two cycles, we have:

$$F_s = -0''0031 \cos(\Omega - \alpha) - 0''0016 \sin(\Omega - \alpha) - 0''0050 \cos(\Omega + \alpha) + 0''0008 \sin(\Omega + \alpha) \quad (2.28)$$

and so from (2.10), we will obtain:

$$N = -0''00124 \pm 0''00018, \quad \delta_N = -0002 \pm 0{,}0002, \quad \beta_1 = \cdot -, \quad \beta_2 = \sim$$

TABLE 19.

R	Phase $\beta-\alpha$	Phase $\beta+\alpha$	M_1	M_2	Phase $\beta-\alpha$	Phase $\beta+\alpha$	M_1	M_2
1	2	3	4	5	6	7	8	9
		0^h				3^h		
0	16^h	17^h	−39	—	13^h	19^h	+26	—
1	15	15	+6	—	12	18	+22	—
2	13	14	+3	—	10	17	−16	—
3	12	13	+16	—	9	15	−19	—
4	11	11	+9	—	8	14	−14	—
5	9	10	+19	—	6	13	−11	—
6	8	9	+6	+6	5	11	+13	+18
7	7	8	−5	−23	4	10	+2	−20
8	6	6	+5	−11	3	9	−22	−23
9	4	5	+16	−6	1	8	+12	−20
10	3	4	+20	+18	0	6	+1	−4
11	2	2	−20	−36	23	5	−7	−16
12	0	1	−11	+1	21	4	−22	+1
13	23	0	−8	−21	20	2	−7	−5
14	22	23	+14	+23	19	1	+18	+23
15	20	21	−3	−2	18	0	−4	+12
16	19	20	−26	+7	16	23	0	+9
17	18	19	+28	+24	15	21	+7	−6
18	17	17	−22	−11	14	20	−8	−10
19	15	16	+31	+20	12	19	−1	+19
20	14	15	−18	+6	11	17	+4	−15
21	13	14	−18	−12	10	16	−34	−15
22	12	12	—	+42	9	15	—	+1
23	10	10	—	+28	7	14	—	+7
24	9	9	—	+3	6	12	—	−3
25	8	8	—	+6	5	11	—	−33
26	7	6	—	−4	4	10	—	−9
27	5	5	—	−19	2	8	—	−11
28	4	4	—	−26	1	7	—	+12
29	3	3	—	+4	0	6	—	+6
30	1	1	—	−9	22	5	—	+1
31	0	0	—	+2	21	3	—	+19
32	23	23	—	−16	20	2	—	+1
33	21	21	—	+6	18	1	—	−11
34	20	20	—	−39	17	23	—	+5
		6^h				9^h		
0	11	23	−22	—	8	2	+14	—
1	10	22	+35	—	7	1	+36	—
2	8	21	−16	—	5	0	−40	—
3	7	19	−22	—	4	22	−25	—
4	6	18	−3	—	3	21	−29	—
5	5	17	−10	—	1	20	−5	—
6	3	16	+3	+6	0	19	−14	−70
7	2	14	−16	−16	23	17	−26	37
8	1	13	−24	−18	22	16	−31	+12
9	23	12	+2	−3	20	15	+4	+20
10	22	10	+32	+28	19	13	+12	+13
11	21	9	+16	+20	18	12	+25	+78
12	20	8	+25	+6	16	11	+26	+54

Table 19, cont'd.

R	Phase ☊ − α	Phase ☊ + α	M_1	M_2	Phase ☊ − α	Phase ☊ + α	M_1	M_2
1	2	3	4	5	6	7	8	9
		6^h				9^h		
13	18	7	−7	−6	15	10	+9	−7
14	17	5	+7	0	14	8	+17	+35
15	16^h	4^h	−3	+1	13^h	7^h	−7	+6
16	14	3	−5	−3	11	6	−30	−15
17	13	1	−10	−7	10	4	−9	+11
18	12	0	−14	−12	9	3	−20	+20
19	11	23	+2	−4	7	2	−5	0
20	9	21	−22	−19	6	0	−36	−70
21	8	20	+4	+5	5	23	−27	−62
22	7	19	—	−11	4	21	—	−17
23	5	18	—	−3	3	20	—	−6
24	4	16	—	+12	2	19	—	−8
25	3	15	—	+3	0	17	—	−29
26	2	14	—	−22	23	16	—	+23
27	0	13	—	+4	22	15	—	+8
28	23	11	—	+11	21	13	—	+28
29	22	10	—	+6	19	12	—	+14
30	20	9	—	+1	18	11	—	−24
31	19	7	—	−2	17	10	—	−31
32	18	6	—	−10	15	8	—	−4
33	17	5	—	−10	14	7	—	+21
34	15	4	—	+3	13	6	—	−21
		12^h			15^h			
0	5	5	+28	—	1	8	−9	—
1	4	3	−8	—	0	7	+11	—
2	3	2	−37	—	23	6	−42	—
3	1	1	−29	—	22	4	+4	—
4	0	23	−7	—	20	3	+21	—
5	23	22	−6	—	19	2	+19	—
6	22	21	−30	−1	18	0	−29	−20
7	20	20	−20	−14	16	23	−3	+14
8	19	19	+14	+24	15	22	−16	+28
9	18	17	+16	+42	14	21	−9	+32
10	16	16	+10	+26	13	19	+49	+37
11	15	15	+8	+7	11	18	+32	+45
12	14	13	−23	+10	10	17	−40	−16
13	13	12	+8	−6	9	15	−35	−27
14	11	11	−4	+34	7	14	−35	−23
15	10	10	+13	0	6	13	−5	−32
16	9	8	−8	−2	5	12	−4	−15
17	7	7	+30	+6	3	10	+5	0
18	6	6	−40	−38	2	9	−33	+5
19	5	4	−41	−24	1	8	−53	−21
20	4	3	−12	−59	0	6	−6	−77
21	2	2	−38	−47	23	5	−11	−9
22	1	0	—	−8	22	3	—	+37
23	0	23	—	−6	21	2	—	+18
24	23	22	—	−6	19	0	—	−20

Table 19, cont'd. /130

R	Phase		M1	M2	Phase		H1	H2
	Ω-α	Ω+α			Ω-α	Ω+α		
1	2	3	4	5	6	7	8	9
		12h				15h		
25	22	20	—	-19	18	23	—	+14
26	20	19	—	+47	17	22	—	+41
27	19	17	—	+4	16	20	—	-9
28	17h	16h	—	+26	14h	19h	—	+11
29	16	15	—	+12	13	18	—	-17
30	15	14	—	+8	12	17	—	+14
31	14	13	—	-34	10	15	—	+6
32	12	11	—	-6	9	14	—	-13
33	11	10	—	+30	8	13	—	+13
34	10	9	—	-15	7	11	—	-37
		18h				21h		
0	23	11	0	—	20	14	+5	—
1	21	10	+7	—	18	12	+24	—
2	20	8	-34	—	17	11	-4	—
3	19	7	+1	—	16	10	-11	—
4	17	6	+33	—	14	8	+21	—
5	16	4	+12	—	13	7	+5	—
6	15	3	+8	+4	12	6	-14	+7
7	14	2	+25	+18	10	5	-6	-6
8	12	0	+14	+6	9	3	-23	-19
9	11	23	-13	-9	8	2	-40	-26
10	10	22	-8	+2	6	0	0	+32
11	8	21	-19	-13	5	23	-33	-46
12	7	19	-5	-8	4	22	-41	-26
13	6	18	-36	-40	3	21	-22	+19
14	5	17	-24	-14	2	20	+2	+6
15	3	16	-21	-27	0	18	-4	+7
16	2	14	-16	-14	23	17	-9	-16
17	1	13	+31	+32	22	15	+5	+24
18	23	12	+14	+14	20	14	+17	-11
19	22	10	-7	-9	19	13	+12	+27
20	21	9	+8	+4	18	12	+21	+25
21	20	8	+7	+8	16	10	-3	+12
22	18	6	—	-6	15	9	—	+19
23	17	5	—	+4	14	8	—	+16
24	16	4	—	+2	12	7	—	-2
25	14	2	—	+4	11	5	—	-2
26	13	1	—	+18	10	4	—	-3
27	12	0	—	-4	9	3	—	-28
28	11	22	—	-11	7	1	—	-4
29	10	21	—	+5	6	0	—	-12
30	8	20	—	-8	5	23	—	-12
31	7	19	—	0	4	22	—	-21
32	5	17	—	-8	2	20	—	+4
33	4	16	—	-6	1	19	—	-7
34	3	15	—	+2	0	18	—	-2

TABLE 20.

Phase	Means of the values of F_s, distributed according to the phases of the argument (in 0".001)				Phase	Means of the values of F_s, distributed according to the phases of the argument (in 0".001)			
	$\Omega - \alpha$		$\Omega + \alpha$			$\Omega - \alpha$		$\Omega + \alpha$	
	R=0-21	R=6-34	R=0-21	R=6-34		R=0-21	R=6-34	R=0-21	R=6-34
0^h	−4	−5	−15	−13	12	+4	+8	+13	+5
1	−7	−13	+4	0	13	+7	+6	+1	+4
2	−20	−13	−10	−7	14	0	+2	−9	−2
3	−13	−9	−6	+4	15	+4	+11	−7	+2
4	−11	−13	+1	−5	16	−1	+8	−4	+5
5	−15	−15	+4	−17	17	+4	+13	−17	+3
6	−16	−19	−11	−4	18	+8	+2	+2	−4
7	−15	−18	+6	−1	19	+7	+15	+5	+4
8	−9	−8	−3	+4	20	−1	+2	−9	−5
9	−15	−4	−8	0	21	+20	+9	−16	−1
10	−8	−5	+7	+11	22	−5	+8	−10	+2
11	−3	+1	+7	+1	23	−9	+2	−10	−14

6.3.10. Possible Case of Approximation of a Half-Year Term, The Correction Values for the Nutational Constant /131

According to our initial plan, we must disregard the periodic term of the change of F_s and then redetermine the corrections in the declinations and proper motions. We return to the matter of the influence of error on the gradual value R in the final results of our calculation. The necessity for this comes from the following considerations.

First, from the method of determination of R that we used, the changes in this quantity can have a false linear behavior; second, we obtained earlier some indications that the value which we found for n ought somehow to be increasing. Now, however, the correction Δn seems to be practically equal to zero. This last result deserves greater attention, because we conclude that the assumed value of n requires correction since it depends on the separate values of the correction in the nutational constant of the right ascension of the pair of an observation of which the correction was obtained. This type of dependence seems to be a 1/2 year term in the values of ΔN, and it was discovered in the results of Przybyllok, Kulikov and Jakson.

Because the previous explanation of this term is not valid any longer, we come to the conclusion that this was caused by some systematic errors in the observations, and it especially seems to be the consequence of the phenomenal linear changes of R. We put the difference between the true gradual value R and the supposed R' in the formula of a series from which we take out the first two terms: /132

$$R - R' = C_0 + C t \tag{2.29}$$

The difference in micrometric readings for the pair with right ascension a can also be put approximately as a linear function of time

$$m = m_0 - \frac{p}{R} t \cos\alpha$$

where p is the precession in the declination. We obtain:

$$m(R - R') = C_0 m_0 + \left(C_1 m_0 - C_0 \frac{p}{R} \cos\alpha\right) t + c t^2$$
$$c = - C_1 \frac{p}{R} \cos\alpha \tag{2.30}$$

For the determination of the nutational constant, the equation of the relation takes the form:

$$x + t y + v \, \Delta N = u \tag{2.31}$$

where x and y are as in (2.25) and u is the coefficient of N_0 in (22). The free term u includes both the arbitrary and systematic errors. Since we are now interested in the influence of the square term only, we use (2.31) together with the following equation:

$$t y + v \, \Delta N' = c t^2 \tag{2.32}$$

in which ΔN' is exactly that part of the correction that is due to the influence of the false linear behavior in the assumed R' values. Then the regular equations take the form

$$[t^2]y + [tv]\Delta N' = c[t^3]$$
$$[tv]y + [v^2]\Delta N' = c[t^2 v] \qquad (2.33)$$

In the computation of the free terms of these equations, and the coefficients of the unknowns, we can substitute an integration for the sum. We will not give the elements in detail, but rather /133 the large transformations which must be done, and we will not give the solutions of (2.33) in a known formula. We will limit ourselves to giving the formula in the special case of the determination of the nutational constant from the observed data of the I.L.S. for the years 1900-1915, i.e.

$$\Delta N' = \frac{1{,}8\, m^2\, \alpha\, C_1}{1 + 0{,}2\cos 2\alpha - 0{,}3\sin 2\alpha - 0{,}3\sin^2\alpha} \qquad (2.34)$$

If we take $C_1 = -0''.003$, we obtain a curve which represents satisfactorily the half-year term which we find in the corrections of the nutational constant found by Przybyllok. Figure 4 shows a graph of the function of the nutational constant in terms of the right ascension of the observed pairs for a linear behavior of R ($C = 0''.001$). As independent variable, we have the right ascension expressed in hours; as dependent variable, we have the correction in the nutational constant in $0''.001$. In the construction of the graph, the scale to the left gives the independent variable (ΔN_c) and that to the right is related to the points representing the values of ΔN_o found by Przybyllok.

Przybyllok found the correction in the nutational constant separately from each station. We obtained the mean value for all the stations, and then we separated the pairs into 12 groups and computed the mean value of ΔN for each group. These mean values are given by dots in Fig. 4.

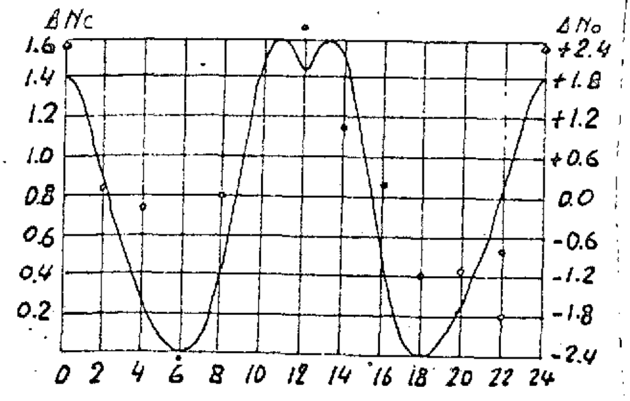


Fig. 4.

These assumptions are /134 not sufficient to show that the real case of the appearance of the half-year term in the value of the nutational constant is an error in the value of R obtained. They only show that such a case is possible. We have already said that it was necessary to examine whether it was probable for the false linear behavior in the changes of R to affect the results of our computations. The method used for the determination of ΔR does not exclude the probability of such behavior. Checking for this is performed as follows: First, by using (2.28), we again put the nutational constant of the latitude from F_s, obtaining the values F_6. These are given in Table 30. Moreover, taking

$$F_6 = x_2 + K\psi_2 + K^2 z_2 \tag{2.35}$$

and using the method of least squares, we obtain the following formula for Z_2

$$Z_2 = 10^5 (p_0 \Sigma F_6 + p_1 \Sigma K F_6 + p_2 \Sigma K^2 F_6) \tag{2.36}$$

The coefficients of this formula are given in Table 21.

TABLE 21.

k	p_0	p_1	p_2
0-21	+375.93	- 98.07	+ 4.45
0-22	+324.67	- 81.16	+ 3.52
0-34	+ 2.12	- 0.35	+ 0.01
6-34	+ 3.33	- 0.40	+ 0.01

We do not give all the values of Z_2. It is satisfactory just to write that no dependence of these values on a has been discovered. It seems that the values of R were obtained almost free of the systematic errors that can be shown on a graph of a function of time. Relating this to the determination of the corrections of declination and special motion in a second approximation, we let out the last term in the right-hand side of (2.35) so that we can use (2.26), substituting F_6 for F_4. This year we obtained the computation in a larger number of figures. The results are given in column 7 of Table 4 and in column 7 of Table 5. Then we compute /135

$$F_7 = F_4 - (x_2 + K y_2)$$

the values of which are given in Table 30. In Fig. 5 we show the consecutive phases in the improvements of the original data and their preparation for harmonic analysis. We take as an example pair 96. F_1 is the direct result of the computation

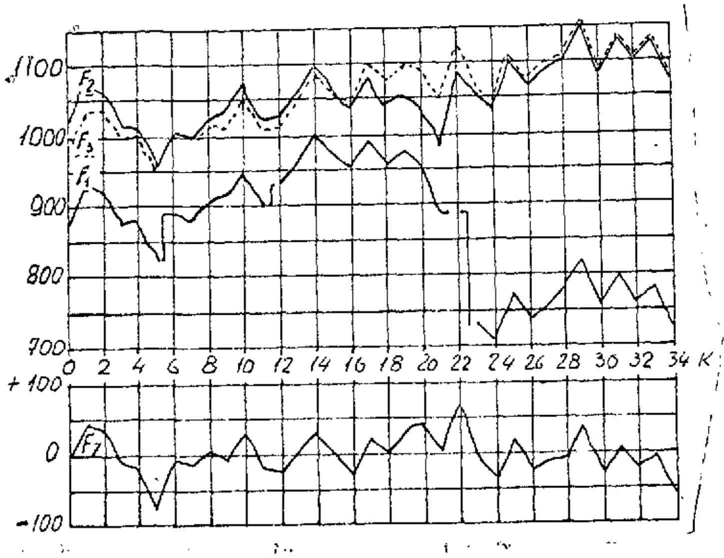


Fig. 5.

from (2.11). F_2 is the result reduced by use of the special values of the initial declination and special motion of the center of the pair through the total period, F_3 the same after the introduction of the correction for the error in R. Finally, F_7 gives the wholly complete data for harmonic analysis. The dependent variable is in 0".001.

6.3.11. Time Diagram of Analysis of the Nonpolar Variable of Latitude /136

From the studies of the nonpolar variable of latitude from observations in the international stations during 1922-34, Kimura discovered an 11 year term.

$$0''.018 \sin\left[\alpha - 32^\circ.43\,(t - t_0) + 86^\circ\right] \qquad t_0 = 1924.058$$

Although Panchenko showed later that this result was not exactly correct, we think that we must consider the matter of its existence again, based on more sufficient observations than those of Kimura. Moreover, it would be important to answer a more general question. Are there some latitude changes of long period and/or of a period larger than 19 years of nutational motion? If such oscillations exist, they can be the cause of systematic errors in the nutational constant, the value of the axis of the ellipse of nutation, and deceleration. There is also the case of the appearance of a half-year term in the phase.

Some methods may possibly discover a hidden periodicity of this kind. Our problem has certain special properties that make the use of some of these methods impossible without some preliminary calculations. After a series of experiments, we ended with the following plan of solutions. Making F_7 for the expression the analogue of (2.6)

$$F_7 = A_1 \cos\mu t \cos\alpha + B_1 \cos\mu t \sin\alpha + A_2 \sin\mu t \cos\alpha + B_2 \sin\mu t \sin\alpha \qquad (2.37)$$

where μ is the frequency related to the required period by the relation $\mu = 2\pi/T$ and placing

$$\left.\begin{aligned} A_1 \cos\mu t + A_2 \sin\mu t &= A \\ B_1 \cos\mu t + B_2 \sin\mu t &= B \end{aligned}\right\} \qquad (2.38)$$

we will have

$$F_7 = A\cos\alpha + B\sin\alpha \qquad (2.39)$$

Because we considered only the long-period latitude change, we can assume as an approximation that the coefficients A and B are constant for a year, and for their determination we use the usual methods of harmonic analysis. For each of the 12 groups of pairs, and for each year, we find the mean value of the quantity F_7. These are denoted by F and are given in Table 22. Thus for each year of observation, we obtain the 12 F values from which we find again for each year separately the coefficients A and B. The computation follows the usual method of harmonic analysis for 12 dependent variables in which we assumed, for the sake of simplicity, that a takes the values 0, 2, 4, ... hours and not 1, 3, 5, ... hours, which it actually takes. So we do not get coefficients A and B, but two other magnitudes A' and B' related to these by the relations /137

$$\left.\begin{aligned} A &= A'\cos 15^\circ - B'\sin 15^\circ = 0{,}996\,A' - 0{,}259\,B' \\ B &= A'\sin 15^\circ + B'\cos 15^\circ = 0{,}259\,A' + 0{,}996\,B' \end{aligned}\right\} \qquad (2.40)$$

The results are obtained initially for the 1900-1921 cycles and 1906-1924 cycles separately, and then they are compiled.

The values for A' and B' are expressed as in F_7 in 0".001 and are given in columns 2 and 7 of Table 23. We use this part of the original data for the determination of the hidden periodicities in nonpolar latitude change.

We will use Fuhrich's method as that which most approximates the conditions of our problem, rather than the usual /138
method of Schuster. Pollak used this method in the linear periodic analysis of the motion of the pole and obtained the /140
formula and plan of calculations from his papers, and especially the computation of the automatic correlation of the coefficients. Similarly, the second coefficients that we have denoted by y"(A)

TABLE 22.

R	Values for the groups											
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
						Cycle 1900-1921						
0	−38	+52	−10	+21	−6	+9	+44	−13	+18	+12	+15	−3
1	−3	+36	+32	+35	+22	−28	+20	+10	+31	+2	+47	+33
2	+1	−24	−17	−22	−33	−49	−34	−39	−28	−24	+17	+1
3	−14	−17	−34	−22	−24	−33	−22	+10	+12	+1	−6	−17
4	+12	−31	−6	−8	−18	−14	+7	+30	+44	+39	+21	−19
5	−3	−32	−18	−5	+12	+18	−12	+31	+23	+10	0	−33
6	+7	+12	+10	−17	−25	−9	−29	−23	+3	+14	−12	−3
7	+1	−12	+3	−29	−22	−11	−12	+2	+28	+18	−1	+9
8	−6	−12	−40	−13	−29	+3	+16	−9	+24	+8	−7	−6
9	+20	+30	−5	−5	+12	+12	+26	−2	−21	−4	−52	+23
10	+31	−6	+30	+25	+23	+8	+14	+55	−13	+6	+10	+13
11	−4	−3	+4	+31	+25	+17	−2	+39	+10	−39	−28	−4
12	−17	−26	+30	+27	+27	−8	−26	−34	+1	−28	−21	−4
13	−9	−7	−2	−8	+22	+28	−4	−29	−48	−20	−10	−2
14	+17	+16	+14	+19	+36	+42	−38	−28	−33	−14	+22	+35
15	−17	−8	+5	−4	+1	+14	+16	+1	−38	+9	+1	−6
16	−28	0	−2	+9	−34	+14	−36	+2	−32	+13	0	−28
17	+41	+10	+3	−15	−4	+23	+55	+11	+11	+46	+4	+42
18	−27	+5	−8	−12	−12	−39	−24	−26	+16	+15	+26	+40
19	+31	+2	+50	+22	+1	−30	−38	−47	−22	+9	+23	+46
20	−26	+32	−14	−34	−34	−2	+16	+12	+5	+20	+43	+5
21	−9	−37	+18	−21	+10	−40	−12	−5	+10	+6	+9	+10
						Cycle 1906-1934						
6	+6	+16	+8	−9	−68	−25	+20	−26	−5	+23	+16	−7
7	−24	−48	−9	−23	−27	−24	+5	+26	+19	+21	+4	−29
8	−10	−12	−32	−8	+26	+21	+34	+16	+13	+15	−3	−20
9	−17	−17	−11	+1	+34	+36	+55	+20	−11	+8	−38	+2
10	−1	−6	+26	+33	−4	+8	+60	+51	+5	+18	+31	+24
11	−31	−12	+4	+39	+71	−5	−49	+77	+12	−36	−50	−39
12	+39	−1	+22	+9	+34	+19	−12	−14	−3	−17	−23	−15
13	+6	−10	−4	−11	+2	+5	−20	−9	−41	−30	+20	−14
14	0	+21	−1	+19	+27	+48	−27	+28	−21	−9	+20	+20
15	−24	+13	+15	−11	+16	+29	−32	−40	−41	−6	0	+10
16	+40	+10	+1	+6	−40	+59	0	+2	−25	+1	−40	+7
17	+15	−10	+20	−24	+23	+31	−36	−22	+23	+44	−20	+35
18	−24	−3	−8	−10	+26	−28	−10	−7	−13	+8	−14	−5
19	−13	+17	0	+10	−28	−13	48	+22	−22	+7	+26	+37
20	−43	−6	−7	−31	−74	−42	−104	−42	−11	+14	+27	+22
21	−51	−16	+27	−17	−70	−30	−47	+11	+6	+4	+10	+15
22	+33	+13	−4	−10	+4	+8	−6	+47	+4	−1	+9	+39
23	+38	+12	+11	−5	+9	−7	+11	+32	+12	−4	+23	+11
24	+25	−7	+9	+18	+8	−3	−2	+7	−6	−2	−2	−4
25	+8	−35	+6	+2	+48	−6	−8	+32	+2	−1	−2	+6
26	+51	−3	−20	−8	+31	+55	+42	+61	+18	+8	−4	−13
27	−24	−13	−8	+17	+10	+10	−12	−2	+7	−22	−29	−17
28	−42	−7	+26	+11	+47	+36	+32	−14	−5	−24	+4	−14
29	−9	−1	−1	+16	+35	+17	+5	23	+2	+2	−23	+6
30	+12	+4	−12	+15	−16	+5	+46	+6	−5	−14	−20	−15
31	+1	+34	−1	+4	−26	−35	0	−5	+2	−6	−5	−32
32	−13	+16	+5	−11	+6	−10	+16	−38	−15	−3	+16	−24
33	+16	−1	−12	+3	+28	+26	+54	−22	+5	−15	−22	+4
34	−35	+4	+4	+25	−38	−8	−17	−35	−3	+2	−19	−25

TABLE 23.

R	A'	$y'(A')$	$y''(A')$	$\mu_a R$	U_a	B'	$y'(B')$	$y''(B')$	$\mu_b R$	U_b
1	2	3	4	5	6	7	8	9	10	11
0	−3	—	—	0°	−15	−8	—	—	0°	−10
1	+13	+46	+77	21	−2	+2	+60	+82	21	+3
2	+26	+30	+51	41	+10	−3	+34	+55	43	+2
3	+3	+21	+33	62	−11	−18	+36	+29	64	−8
4	−1	+10	+7	82	−12	−28	+18	+8	86	−15
5	−16	−2	−19	103	−27	−13	−4	−20	107	−1
6	+18	−22	−42	124	+13	−5	−33	−50	128	+7
7	+1	−36	−61	144	+3	−21	−54	−72	150	−13
8	−11	−51	−79	165	−9	−15	−56	−82	171	−13
9	−14	−67	−85	185	−6	+7	−48	−75	193	+9
10	−5	−49	−76	206	+6	−2	−23	−66	214	−7
11	−31	−31	−46	227	−21	+13	−32	−48	235	+3
12	−1	−22	−21	247	+9	+20	−11	−10	257	+7
13	+1	−12	+2	268	+8	+14	+29	+24	278	+1
14	+7	−9	+32	288	+8	+17	+31	+45	300	+5
15	+1	−13	+63	309	+2	+11	+39	+54	321	+3
16	+2	+25	+77	330	−4	+6	+50	+67	342	−2
17	+8	+67	+84	350	−4	−8	+68	+72	4	−10
18	+8	+58	+80	11	−4	−7	+57	+61	25	−2
19	+30	+35	+53	31	+15	+7	+35	+34	47	+17
20	+25	+9	+41	52	+11	−18	+12	+8	68	−8
21	+13	+20	+20	73	−1	−12	+8	+5	89	+1
22	+10	+13	−4	93	−1	−7	+30	−27	111	+5
23	+9	−6	—	114	+4	−4	+19	—	132	+8
24	+4	−31	—	134	−1	+7	−21	—	154	+15
25	−8	−4	—	155	−6	0	−38	—	175	+2
26	−21	−40	—	176	−13	−8	+1	—	196	−13
27	−13	−58	—	196	−2	+8	+16	—	218	+3
28	−23	+1	—	217	−12	+16	+10	—	239	+6
29	−7	—	—	237	+3	+11	—	—	261	−2
30	−10	—	—	258	−3	+3	—	—	282	−10
31	+9	—	—	279	+16	+3	—	—	303	−9
32	+4	—	—	299	+5	+7	—	—	325	−1
33	−13	—	—	320	−19	+9	—	—	346	+7
34	+3	—	—	340	−3	+7	—	—	8	+5

and y"(B) can be so understood as being of sinusoidal form that the frequency of the largest nutations are accurately determined. For A' this was found to be equal to 20°6 per year; for B', 21°4 per year. These correspond to periods of 17.5 and 16.8 years.

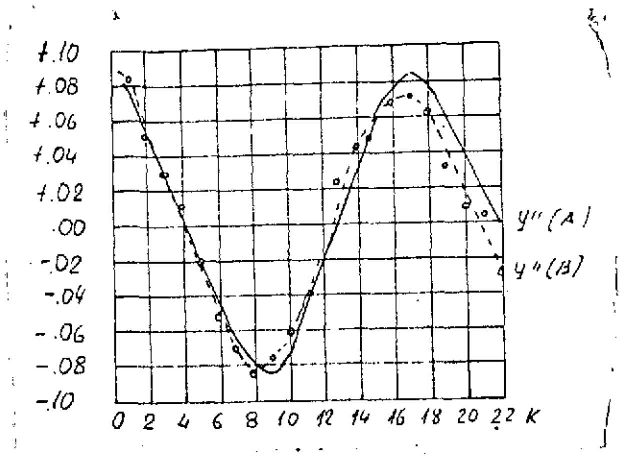


Fig. 6.

In order to discover other periods, we start again from the data of the largest periodic terms. Thus we put A' and B' in the form

$$A' = m_1 \cos \mu_a R + m_2 \sin \mu_a R + v_a$$

$$B' = n_1 \cos \mu_b R + n_2 \sin \mu_b R + v_b$$

where $\mu_a = 20°6$, $\mu_b = 21°4$. The values of the arguments $\mu_a R$ and $\mu_b R$ are given in columns 5 and 10 of Table 23. After this, we find by harmonic analysis the following values of the coefficients of the periodic terms.

$$m_1 = 0.''0021 \qquad m_2 = -0.''013$$

$$n_1 = 0.''0096 \qquad n_2 = 0.''0086$$

and finally the values of the remaining u_a and u_b that are given in Table 23, columns 6 and 11, in 0".001. These seem to be arbitrary, so that the continuation of the linear periodic analysis would make no sense. Thus we have succeeded in the discovery only of a nutation of period near 17 years. Although Fuhrich's method does not allow us to compute rigorously the precision of the values obtained, an error of 1 or 2 years is very probable. Hence we can recognize the oscillation taken together with the nutational term with a period of 18.6 years. So we have the justification for our reasoning, i.e. that other terms of long period in (2.37) including the 11-year term do not appear in nonpolar latitude change. /141

6.3.12. Determination of ΔN, Δn, β_1 and β_2 (β' Approximation)

So far we have used A' and B' which were acceptable for the study of periods. For the determination of the coefficients in (2.6), it is necessary that we use A and B computed from (2.40). The results are given in Table 24 in columns 3 and 4 for 1900-1921 and in columns 7 and 8 for 1906-1934. The rest of the columns give the remaining u_a and u_b (in 0".001).

TABLE 24.

K	☊	1900-1921 A	1900-1921 B	1900-1921 V_α	1900-1921 V_b	1906-1934 A	1906-1934 B	1906-1934 V_α	1906-1934 V_b
1	2	3	4	5	6	7	8	9	10
0	250°	− 5	− 8	−14	− 3	—	—	—	—
1	232	+ 11	+ 2	+ 2	+ 11	—	—	—	—
2	212	+ 23	− 3	+ 15	+ 10	—	—	—	—
3	193	+ 1	− 18	− 6	− 3	—	—	—	—
4	173	− 3	− 28	− 7	− 13	—	—	—	—
5	154	− 18	− 13	− 20	+ 1	—	—	—	—
6	135	+ 15	0	+ 17	+ 11	+ 18	− 10	+ 22	− 2
7	115	+ 7	− 16	+ 11	− 10	− 10	− 26	0	− 21
8	96	− 5	− 18	+ 2	− 16	− 22	− 11	− 8	− 9
9	77	− 2	+ 14	+ 6	+ 10	− 30	+ 1	− 13	− 1
10	57	− 4	+ 5	+ 5	− 3	− 10	− 8	+ 8	− 13
11	38	− 18	+ 13	− 9	+ 1	− 48	+ 13	− 31	+ 5
12	19	− 4	+ 21	+ 4	+ 7	− 2	+ 20	+ 12	+ 10
13	359	− 5	+ 18	0	+ 3	+ 4	+ 9	+ 13	− 1
14	340	+ 12	+ 23	+ 14	+ 9	− 2	+ 10	+ 2	0
15	321	− 7	+ 4	− 7	− 8	+ 5	+ 18	+ 3	+ 8
16	301	0	+ 2	− 4	− 6	0	+ 10	− 8	+ 4
17	282	+ 4	− 13	− 2	− 16	+ 8	− 3	− 5	− 6
18	263	+ 14	− 15	+ 6	− 13	− 2	+ 1	− 18	+ 2
19	243	+ 36	+ 16	+ 27	+ 23	+ 21	− 3	+ 3	+ 1
20	224	+ 7	− 22	− 2	− 11	+ 39	− 15	+ 21	− 8
21	205	+ 7	− 9	− 1	+ 5	+ 15	− 15	0	− 6
22	186	—	—	—	—	+ 8	− 7	− 3	+ 3
23	167	—	—	—	—	+ 7	− 4	+ 1	+ 6
24	147	—	—	—	—	+ 2	+ 7	+ 2	+ 16
25	128	—	—	—	—	− 10	0	− 4	+ 7
26	109	—	—	—	—	− 23	− 8	− 12	− 4
27	89	—	—	—	—	− 15	+ 8	+ 1	+ 8
28	76	—	—	—	—	− 25	+ 16	− 7	+ 13
29	51	—	—	—	—	− 9	+ 11	+ 9	+ 5
30	31	—	—	—	—	− 12	+ 3	+ 4	− 6
31	12	—	—	—	—	+ 7	+ 3	+ 20	− 7
32	353	—	—	—	—	+ 2	+ 7	+ 10	− 3
33	333	—	—	—	—	+ 15	+ 9	− 13	− 1
34	314	—	—	—	—	+ 1	+ 7	− 3	− 1

Based on the results of the previous paragraph, we put

$$A = A_1 \cos \delta\!\!\!B + A_2 \sin \delta\!\!\!B + u_a \quad , \qquad B = B_1 \cos \delta\!\!\!B + B_2 \sin \delta\!\!\!B + u_b \qquad (2.41)$$

and by the methods of harmonic analysis, we find the coefficients A_1, A_2, B_1 and B_2 and then the remaining terms u_a, u_b. Thus we obtain

1900-21

$$A_1 = -0\overset{''}{,}0053 \pm 0\overset{''}{,}0034 , \qquad A_2 = -0\overset{''}{,}0073 \pm 0\overset{''}{,}0034$$

$$B_1 = +0\overset{''}{.}0105 \pm 0\overset{''}{.}0021 , \qquad B_2 = -0\overset{''}{.}0007 \pm 0\overset{''}{.}0021$$ /143

1906 - 34

$$A_1 = -0\overset{''}{,}0097 \pm 0\overset{''}{,}0031 \qquad A_2 = -0\overset{''}{,}0155 \pm 0\overset{''}{,}0031$$

$$B_1 = +0{,}0105 \pm 0\overset{''}{,}0021 \qquad B_2 = -0\overset{''}{,}0007 \pm 0\overset{''}{,}0021$$

From this, using (2.10), we find for 1900-1921

$$\Delta N = 0\overset{''}{,}051 \pm 0\overset{''}{,}0039 \qquad \Delta\eta = 0{,}0004 \pm 0{,}0005 \quad \beta_1 = 2\overset{'}{,}6 \pm 1\overset{'}{,}7 \qquad \beta_2 = 0\overset{'}{,}0 \pm 1\overset{'}{,}7$$

and for 1906-1934

$$\Delta N = 0{,}0105 \pm 0\overset{''}{,}0021 \qquad \Delta\eta = -0{,}0008 \pm 0\overset{''}{,}0003 \quad \beta_1 = 3\overset{'}{,}8 \pm 1\overset{'}{,}2 \qquad \beta_2 = 0\overset{'}{,}2 \pm 0\overset{'}{,}7$$

These values agree with those given in Chapter 7.3.9. From these results, we are led to the conclusion that the delay in phase is special in the nutation of latitude. Also at the same time, these results give certain indications of the general turning of the ellipse of nutation. Usually in the construction of this ellipse we use a system of Cartesian coordinates on a plane touching the celestial sphere at the mean pole of the Earth, which is also the coordinate origin.

The axis OY has its direction along the mean equinoctial colure from the pole of the ecliptic and OX has direction along the colure of the celestial equator toward the vernal equinox. Then the equation of the ellipse of nutation can be put in parametric form

$$x = nN\cos\varphi\ , \quad y = N\sin\varphi \quad \acute{o}\pi o\upsilon \quad \varphi = 90 + \delta$$

Because we have reformed the usual expression (2.2) to (2.3), it is necessary to substitute the following expression for the previous one:

$$x = nN\cos(\varphi - \beta_1)\ , \quad y = N\sin(\varphi - \beta_2) \tag{2.42}$$

From there, using the formulas of analytic geometry, we find the value of the angle between OY and the longest principal axis of the ellipse of nutation

$$\nu = -\frac{n}{1-n^2}(\beta_1 - \beta_2)$$

If we substitute the values found for the constants

/144

$$\nu = -6'.3 \pm 2'.4$$

Hence, the positive end of the large axis of the ellipse of nutation is divided by the equinoctial colure by 6'8 toward the vernal equinox.

The theory of the rotation of the Earth has not yet given any indications of such a turning of the axis of the ellipse of nutation. This gives rise to some doubt about the results obtained above. We think, however, that it will be useful to keep this result for a further development of the theory, based on our new data, for the internal structure of the Earth. Substituting the arithmetical values in (2.8), we obtain

$$\Delta\delta = -6''.850\cos\alpha . \sin(\delta - 3'8) + 9''.198\sin\alpha\cos\delta \tag{2.44}$$

6.3.13. Comparison with the Results of Other Authors

The new value that we found for the nutational constant is one of the results of the solution of more general problems from the determination of the coefficients of the main terms of nutation and the delay (difference) in phase. Until now, the theoretical value of the proportion of the axis of the ellipse of nutation n had always been used and the initial phase β_1 and β_2 had been taken to equal zero. Thus there does not exist immediate benefit from a detailed development of the results obtained on the basis of the hypotheses mentioned. So we give below a sum of the values of the constants of nutation for comparison.

Table of Results

In connection with these results, we find it necessary to make these notes. Newcomb's value for the nutational constant is the mean value of the 27 values found by other authors. Of these ten, and not one, as was incorrectly stated by Idelson, are smaller than Przybyllok's value, and five are smaller than ours. The accuracy that Kulikov claims his values have for N is overestimated, because in order to find this value, he considered results of different methods from the same original information as independent of each other. /145

T. Hattori used the same original data for the determination of the nutational constant which we used for the solution of most problems (of the general problem). He made his work known only after our computations, which agreed satisfactorily in the choice of methods of solution and in some special problems arising in the reduction of the original data. For example, Hattori gave his latitude in the CG system, and in the same formula (1.1), and found the nonpolar variable of the latitude in common for the three stations. But this is exactly that magnitude we have denoted by F_1.

Hattori did not include the slow changes in latitude common for all pairs, and he did not try to determine the errors in the gradual value. But we saw that sometimes this fact becomes apparent. We also noted that in four cases the corrections in proper motions do not agree with those of Hattori. For this reason, we give the values of $\Delta\mu$ (the ones found by us are in parentheses).

Number of pairs	Epoch		
24	1928.0	$-0''.003$	$(-0''.001)$
35	1903.0	$-0''.022$	$(-0''.016)$
72	1909.0	-0.004	(0.004)
86	1915.0	0.008	$(0''.003)$

For the determination of the constant of nutation, Hattori used various methods. /146

In Chapter 7.3.11, a first step was made toward a separate determination of the main nutational terms. Later Orlov changed the determination of these coefficients from the material of the observations at Pulkovo with a zenith telescope from 1915-1928, and he found that no correction is necessary. Orlov's research gave rise to some doubts at the beginning. As initial conditions he used the instantaneous values of latitude given by Koral. For the determination of the correction for the graduated value, Koral applied -- as we did -- the method of comparison of latitudes which are taken separately for pairs of positive and negative zenith distance. But in the pairs selected, he did not take into consideration obtaining results which are generally free from the probability of errors in the constants of the nutational terms. This could lead to the conclusion of weakness in the 19th term. Actually, it seems that this does not occur. Although in a few cases, the change in the mean values of cos a and sin a for groups of opposite sign zenith distances was taken into consideration in the estimated values, these change from year to year, and [illegible] systematically the same as the change of the sign.

The only effort toward investigating the subject of the existence of phase difference in nutation was done by Morgan. He analyzed the observations of parabolic stars in Washington from 1903 to 1925; he used equations of the following form:

$$\Delta\delta = -(0{,}037\, a' \sin \Omega + b' \cos \Omega)\, \Delta N + (b' \sin \Omega - 0{,}037\, a' \cos \Omega)\, N \Delta\vartheta$$

where a' and b' are constants of reduction. (2.6) has such a form if we substitute in it (2.7) and take

$$\Delta n = 0 \quad , \quad \beta_1 = \beta_2 = -\Delta\vartheta$$ /147

From the observations of declination, Morgan found that $\Delta\theta = 0'.0 \pm 4'.8$ and from the right ascension observations that $\Delta\theta = 13'.2 \pm 3'.6$. The difference between these two values together with the mean error of each one of them gives rise to doubts of the reality of Morgan's results. Morgan himself had the same doubts, so he did not publish the results of his computations and restricted his announcement of them.

The constant of precession, the ratio of the masses of the Earth and the Moon and the constant of nutation are connected as is known by one certain relation which is determined by the theory of rotation of the Earth. By using this relation, we are able to find the theoretical value of N if we assume that the two other constants are known. The value determined this way is greater than the one determined by the observations. We can also note that the new value is different from the theoretical value by a quantity two times greater than the value 9".210 which was taken in Paris at the 1896 meeting and which has been applied up to now in the reduction of the apparent position.

6.4. General Remarks /149

The chapters discussing the examination of the observations cover essentially two parts and deal with two phenomena. The development of all the observations which are concerned with polar motion would also indicate the influence of this phenomenon on other phenomena. This would be very useful for those who investigate polar motion, because they probably could get results which are not yet known and because the research up to the present would be applied, and this is the objective purpose of an applied science. We are not going to investigate these in detail. We shall, however, give a brief summary.

Astronomical data are composed of the "ancient" observations (1000 B.C. - 0), the new ones (1680-1950) and the modern (1950 - present). We separate the last ones because of their homogeneity which is necessary for statistical study. It is important to mention that astronomy was developed by the Babylonians and the Toltecs (ancient people from South America).

The Greeks gave the first scientific basis for research, while the previous peoples continued the development of astronomy with the help of astrology, sorcery and prediction. However, the Toltecs had the most complicated and the most accurate calendar in history, more accurate than the Gregorian.

The history of polar research appears to be large and complicated. The main result (Munk) is that the problem does not have a solution, but is rather a progressive approximation.

From indications of dynamics and reology, it seems that /150
the easiest way is to determine satisfactory forces which act on the Earth and thus predict the future positions of the pole. Moreover, in order to solve the problem of the pole, we must take into consideration many other factors, such as the deformation of the planets and the Moon, the solar wind and magnetohydrodynamic phenomena of the outer portion of the atmosphere due to this wind, the clusters of magnetic fields, and all the phenomena referred to already in various chapters. It is the resultant of the other sciences.

TABLE 30.

R	Ω-α	Ω+α	F_1	Dδ	F_2	F_3	F_4	F_5	F_6	F_7
1	2	3	4	5	6	7	8	9	10	11
					Pair 1					
6	8^h	9^h	+882	+118	+989	+995	+990	+8	+12	+6
7	7	7	856	112	962	958	962	-22	-14	-24
8	6	6	897	106	999	971	979	-8	+3	-10
9	4	5	931	100	1031	967	975	-15	-3	-17
10	3	3	928	94	1028	984	992	0	+13	-1
11	2	2	892	88	986	956	964	-30	-18	-31
12	1	1	929	154	1089	1030	1037	+40	+51	+39
13	23	0	920	156	1083	1002	1007	+7	+12	+6
14	22	22	927	158	1090	1002	1003	+1	+3	0
15	21	21	911	160	1071	985	981	-23	-25	-24
16	19	20	980	162	1135	1055	1048	+41	+34	+40
17	18	18	940	164	1101	1030	1025	+15	+4	+15
18	17	17	875	166	1039	978	988	-24	-36	-24
19	16	16	859	168	1021	974	1001	-13	-27	-13
20	14	15	800	170	964	928	974	-43	-54	-43
21	13	13	761	172	937	912	969	-51	-62	-51
22	12	12	737	268	1007	1005	1055	+33	+25	+33
23	11	11	757	260	1023	1022	1062	+38	+33	+38
24	9	9	754	252	1014	1015	1051	+24	+26	+25
25	8	8	748	244	999	1001	1037	+7	+12	+8
26	7	7	804	236	1045	1047	1082	+50	+58	+51
27	5	6	744	228	974	977	1009	-25	-14	-24
28	4	4	744	220	961	964	994	-43	-29	-42
29	3	3	787	212	995	998	1030	-10	+3	-9
30	1	2	818	204	1016	1019	1053	+11	+21	+12
31	0	0	813	196	1003	1005	1043	-1	+7	+1
32	23	23	803	188	982	990	1032	-15	-10	-13
33	22	22	841	180	1019	1021	1064	+14	+16	+16
34	20	21	+801	+172	+971	+972	+1015	-37	-41	-35
					Pair 2					
0	16	17	+867	+106	+971	+939	-909	-60	-57	-47
1	15	15	909	104	1007	984	954	-18	-21	-9
2	13	14	910	102	1004	989	959	-17	-22	-12
3	12	13	878	100	977	969	939	-40	-48	-39
4	11	11	930	98	1027	1023	995	+12	+2	+9
5	9	10	890	96	983	982	963	-23	-32	-30
6	8	9	963	96	1048	1050	1045	+56	+60	+60
7	7	8	904	94	992	991	995	+2	+9	+7
8	6	6	936	92	1024	1020	1028	+32	+43	+38
9	4	5	933	90	1023	1017	1025	+25	+37	+33
10	3	4	939	88	1033	1032	1040	+37	+49	+46
11	2	2	891	86	983	984	992	-14	-2	-4
12	0	1	941	65	1012	1020	1027	+17	+25	+6
13	23	0	934	60	1001	1021	1026	+13	+18	+1
14	22	23	913	55	973	1006	1007	-10	-8	+3
15	20	21	929	50	979	1025	1021	+1	-3	+15
16	19	20	889	45	927	986	979	-44	-51	-29
17	18	19	957	40	994	1065	1060	+33	-23	+49
18	17	17	860	35	893	975	985	-45	-57	-28
19	15	16	886	30	910	1002	1029	-5	-17	+14
20	14	15	827	25	846	948	994	-43	-54	-23
21	13	14	+809	+20	+833	+944	+1001	-39	-49	-18

On the other hand, research on the phenomenon gives information on the original shape of the Earth, on the surface distribution of the solid mass, on the magnetic field of the Earth, the elasticity or plasticity of the Earth, and on the inner part of the Earth and other things. According to this, we can estimate the extent of research which remains to be done, and sciences like reology, theoretical mechanics, paleontology, and others, from which we will ask information and to which we will give suggestions, will be utilized.

7. GENERAL INTRODUCTION TO THE THEORY OF ELASTICITY /151

7.1. A General Introduction to the Theory

The presentation of the phenomenon and the method of observation constituted the first part of the present paper. The theory will be the second part. From a more general point of view, we could say that in the first part we followed the analytical procedure, where we obtained our conclusions from the observations made. In the second part, we will follow the synthetic procedure, i.e. we will try to find that model of the Earth from which it is possible to predict, according to physical laws, the effect of the influence of physical forces on the Earth in such a way as to agree with the observations.

There are different theories, all of them based on the behavior of the Earth under the action of the rotational forces as well as of other forces due to different phenomena (meteorologic, etc.). In general, a body can be considered as rigid, which never happens, or plastic or elastic. If the deformations are very small, it can be considered as rigid. This is the case with the Earth, and for this reason the first complete theory which was developed assumes that the Earth is a rigid body. The more recent theory, which will be presented in detail, assumes that the Earth is an elastic body obeying Hook's law. Even today, this theory is not completely satisfactory, and new considerations are under research that will be referred to. Because the theory is based on the elasticity of the Earth, it was necessary at the beginning to present a few points about the theory of elasticity. This was considered necessary because of the future use of different concepts, and because of the fact that in this way we will have a more general point of view for our study.

Throughout the presentation of the theory, tensor analysis /152
is employed; therefore, it was necessary also to give an introduction to the methods of tensor calculus. We also tried to combine the theory of elasticity with tensor calculus, which was necessary for compactness of presentation. All these were presented in detail, taking for granted that the readers would be faced with the same difficulties which faced the writer initially.

The development was also based on the concepts which will be used below.

7.2. Introduction to Tensor Analysis

Tensors play an important role in physics, including the general theory of relativity and the electromagnetic theory. Also, one of the most important applications of tensors is to anisotropic solids.

As an initial example, we will consider the flow of electric current. Ohm's law can be written as

$$\bar{j} = \bar{\sigma}E \quad (1)$$

where $\bar{j}$ is the electric density, $\bar{E}$ the electric field, and σ is the conductivity. If the medium is isotropic, then the conductivity σ is a scalar quantity, and for the x component, we have, for example,

$$j_1 = \sigma E_1 \quad (2)$$

But if the medium is anisotropic, as happens in many crystals, the electric density may depend on the electric fields of all three directions. Considering a linear relation, we can write Eq. (2) as: /153

$$j_1 = \sigma_{11} E_1 + \sigma_{12} E_2 + \sigma_{13} E_3 \quad (3)$$

and in general

$$\boxed{j_i = \sum_k \sigma_{ik} E_k} \quad (4)$$

For three-dimensional space, the scalar conductivity σ is given by a set of nine elements, σ_{ik}

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

This set of nine elements forms a tensor. For any case, a set of nine elements does not necessarily represent a tensor, and a proof is needed in order to conclude that it is a tensor. Therefore, a tensor is used for quantities which need nine functions for their definition in space (i.e., when we take account of many influences).

A quantity which does not change with a rotation of the coordinate system, i.e. which is invariant, is called a scalar. A quantity whose components transform according to the formulas:

$$x_1' = a_{11} x_1 + a_{12} x_2$$

$$x_2' = a_{21} x_1 + a_{22} x_2$$

(like the components of the distance of a point from the origin) is called a vector. We observe that a different definition of a vector is given from that usually known, based on the transformation of the components according to definite formulas of rotation. Exactly this property of transformation is adopted as the main characteristic of the definition of a vector. In the above definition, we can either use the formula of the transformation of the components (or computation of differentials → contravariant vector) or the formula of the transformation of basis (or computation of partial derivatives → covariant). The difference is that we have either $\partial x_i'/\partial x_j$ or $\partial x_j/\partial x_i'$. /154

Since in a Cartesian coordinate system the two expressions are identical, we will have

$$\frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'} = a_{ij}$$

where $a_{i\lambda}$ are the directional cosines.

The tensor is a general concept. Thus, a scalar quantity is considered a tensor of zero rank, and a vector, a tensor of first rank. Tensor representation is required for quantities which, in order to be defined, need n^m arithmetic quantities, called tensor components, where n = number of spatial dimensions, m = rank of the tensor. A tensor is a quantity independent of the reference coordinate system whose components transform according to a definite transformation law when the coordinate system is changed, and this transformation of the components does not imply a change of the tensor itself, but comes merely as a result of change of the reference coordinate system. In the

Cartesian system, we define a tensor through a transformation relation of the form

$$\boxed{A'_{ij} = \sum_{k,\ell} a_{ik}\, a_{j\ell}\, A_{k\ell}}$$

where $\quad a_{ik} = \frac{\partial x_k}{\partial x'_i}\ , \quad a_{j\ell} = \frac{\partial x_\ell}{\partial x'_j}$

A_{ij} = derived components, $A_{k\ell}$ = given, $a_{ik}\ a_{j\ell}$ = transformation law for rotation, e.g. cosines; see Fedorov 6.7.

The above expression is a generalization of the previously given formula.

Application

Given the matrix $T = \begin{pmatrix} -xy & -y^2 \\ x^2 & xy \end{pmatrix}$, we will examine whether it is a tensor. According to the above formula, we will have

$$T'_{11} = \sum_{\substack{k,\ell \\ k=1,2 \\ \ell=1,2}} a_{1k}\, a_{1\ell}\, T_{k\ell}$$

If the rotation is given by θ, we must have $T'_{11} = -x'y'$. We will find out if this component obeys the law of transformation. In terms of the unrotated coordinates, we have /155

$$T'_{11} = -x'y' = -(x\cos\vartheta + y\sin\vartheta)(-x\sin\vartheta + y\cos\vartheta) =$$

$$= -y\cos^2\vartheta . xy - \cos\vartheta . \sin\vartheta y^2 + \sin\vartheta\cos\vartheta x^2 + \sin^2\vartheta . xy =$$

$$= a_{11}\, a_{11}\, T_{11} + a_{11}\, a_{12}\, T_{12} + a_{12}\, a_{11}\, T_{21} + a_{12}\, a_{12}\, T_{22}$$

(For regular position $\begin{smallmatrix} j \to x \\ j \to y \end{smallmatrix}$ change $\rightleftarrows$ change, for example, $12 \to x^2$, $21 = y^2$). The a are the directional cosines for a Cartesian system. We therefore have an identity; thus, the matrix is a tensor.

We have thus far defined a tensor and have shown how the definition is used in order to find whether a quantity is a tensor or not. The following properties also come from the transformation law:

1. If at a point the components of a tensor are all zero with respect to a coordinate system, then they are also zero with respect to any other coordinate system.

2. If the components of a tensor are identically zero with respect to a coordinate system, they will be identically zero with respect to any other system.

The above properties are fundamental for tensor analysis. Thus, we try to express the laws by equating two tensors or by equating to zero one tensor, because then the change of the coordinate system does not change the expression of a law or of a property. Therefore, during the analysis of these properties, we try to construct a tensor of appropriate order and to select an appropriate coordinate system. (The relativity theory of Einstein is based on the above.)

7.3. Introduction to the Theory of Elasticity /156

In a body in equilibrium under the action of external forces, internal forces are developed which keep in equilibrium every infinitesimal element of the body. By changing the cross section under consideration, the internal forces acting on it also change. They also change in direction and position, depending on the position and direction of the infinitesimal element on which they act. The internal force per unit area $\bar{s}$ is called stress. We analyze the stress in a component σ normal to the infinitesimal element, called "normal stress," and a component τ tangent to the infinitesimal element, called "shearing stress."

Since our object is the Earth, the introduction to the theory of elasticity will be given in a form appropriate to our object of interest and concepts which will be used below will be mentioned.

The angular momentum of a body with respect to an arbitrary point O is the sum of the angular momentums of all material points Σ_i of the body with respect to O. The angular momentum $\bar{G}_i$ of any point Σ_i of mass m_i is given by:

$$\bar{G}_i = \bar{r}_i \wedge m_i \cdot \bar{v}_i \qquad (1)$$

The velocity $\bar{v}_i$ in terms of the velocity of the point O and the angular velocity will be

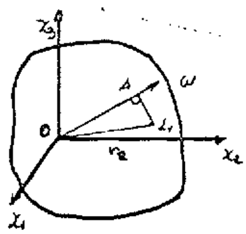
(2)

$$\bar{v}_i = \bar{v}_{(0)} + \bar{\omega} \wedge \bar{r}_i \tag{2}$$

The angular momentum $\bar{G}$ of the body is the sum of the angular momentums G_i of all its material points.

$$\bar{G} = m\bar{r}_g \wedge \bar{v}_0 + \Sigma m_i \left[\bar{r}_i^2 \omega - (\bar{r}_i \bar{\omega}) \bar{r}_i \right] \tag{3}$$

where $m\bar{r}_g = \Sigma m_i \bar{r}_i$



We can express the angular velocity $\bar{\omega}$ as well as the distances $\bar{r}_i$ in terms of their components with respect to the three axes. If $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ are the unit base vectors /157

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \omega_3 \bar{e}_3 \qquad \bar{r}_i = x_{i1} \bar{e}_1 + x_{i2} \bar{e}_2 + x_{i3} \bar{e}_3$$

Substituting in Eq. (3), we obtain

$$\begin{aligned} \bar{G} = m\bar{r}_g \wedge \bar{v}_0 + \Sigma m_i \big[&(x_{i1}^2 + x_{i2}^2 + x_{i3}^2)(\omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \omega_3 \bar{e}_3) - \\ &- (x_{i1}\omega_1 + x_{i2}\omega_2 + x_{i3}\omega_3)(x_{i1}\bar{e}_1 + x_{i2}\bar{e}_2 + x_{i3}\bar{e}_3) \big] \end{aligned} \tag{4}$$

Introducing the moments of inertia

$$J_{11} = \Sigma m_i (x_{i2}^2 + x_{i3}^2) \qquad J_{22} = \Sigma m_i (x_{i3}^2 + x_{i1}^2), \qquad J_{33} = \Sigma m_i (x_{i1}^2 + x_{i2}^2)$$

and the products of inertia

$$J_{12} = J_{21} = -\Sigma m_i x_{i1} x_{i2}, \quad J_{23} = J_{32} = -\Sigma m_i x_{i2} x_{i3}, \quad J_{31} = J_{13} = -\Sigma m_i x_{i1} x_{i3}$$

relation (4) takes the form (5):

$$\begin{aligned} \bar{G} = m\bar{r}_g \wedge \bar{v}_0 &+ (J_{11}\omega_1 + J_{12}\omega_2 + J_{13}\omega_3)\bar{e}_1 + (J_{21}\omega_1 + J_{22}\omega_2 + J_{23}\omega_3)\bar{e}_2 + \\ &+ (J_{31}\omega_1 + J_{32}\omega_2 + J_{31}\omega_3)\bar{e}_3 \end{aligned} \tag{5}$$

Equation (5) gives the angular momentum of a body in terms of the moments and products of inertia with respect to the system $Ox_1x_2x_3$ and of the projections of the instantaneous angular velocity with respect to this system. If point O coincides with the center of mass $\rightarrow \bar{r}_\phi = 0$, and if point O does not move $\rightarrow \bar{u}_{(o)} = 0$. In these cases $\Rightarrow$

$$\Rightarrow \bar{G} = (\lambda_{11}\omega_1 + \lambda_{12}\omega_2 + \lambda_{13}\omega_3)\bar{e}_1 + (\lambda_{21}\omega_1 + \lambda_{22}\omega_2 + \lambda_{23}\omega_3)\bar{e}_2 + (\lambda_{31}\omega_1 + \lambda_{32}\omega_2 + \lambda_{33}\omega_3)\bar{e}_3 \tag{6}$$

If the projections of the vector $\bar{G}$ in our system are

$$\bar{G}_1,\ \bar{G}_2,\ \bar{G}_3 \rightsquigarrow \bar{G} = G_1\bar{e}_1 + G_2\bar{e}_2 + G_3\bar{e}_3 \tag{7}$$

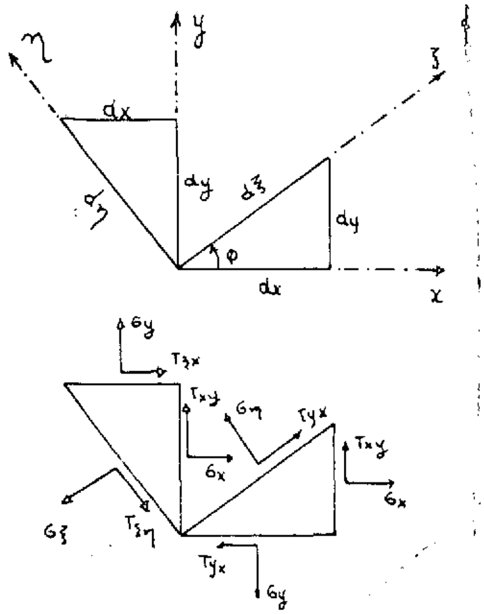
From (6) and (7), we obtain the linear system:

$$\begin{aligned} G_1 &= \lambda_{11}\omega_1 + \lambda_{12}\omega_2 + \lambda_{13}\omega_3 \\ G_2 &= \lambda_{21}\omega_1 + \lambda_{22}\omega_2 + \lambda_{23}\omega_3 \\ G_3 &= \lambda_{31}\omega_1 + \lambda_{32}\omega_2 + \lambda_{33}\omega_3 \end{aligned} \tag{8}$$

The entries of the matrix of the coefficients of ω_1, ω_2, ω_3 of system (8) are the components of a symmetric tensor $\bar{I}$ of second rank ($\lambda_{i\lambda} = \lambda_{ji}$) which is called the inertia tensor. Therefore, the angular momentum with respect to the center of mass /158
or to the fixed point will be

$$\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \text{ or } \bar{G} = I\bar{\omega} \tag{9}$$

We also state without proof that if the inertia tensor is given for a point O of the body with respect to a system $Ox_1x_2x_3$, we can always find a new coordinate system $Ox'_1x'_2x'_3$ with respect to which the inertia tensor is diagonal. That means that the products of inertia are zero and the diagonal elements are the moments of inertia with respect to the three axes Ox'_1, Ox'_2, Ox'_3. The directions Ox'_1, Ox'_2, Ox'_3 are the principal directions



and the characteristic roots of I (eigenvalues) are the principal moments of inertia.

We will now examine the stress state, first in the plane. The plane stress state at a certain position (x,y) of the disk is completely defined by the stresses σ_x, τ_{xy}, σ_y, τ_{yx} of two cross sections normal to the axes x,y. Given the above quantities, we can obtain the stresses σ_ξ, $\sigma_{\xi\eta}$, σ_η, $\sigma_{\eta\xi}$ acting on cross sections parallel to the axes ξ,η of a new coordinate system rotated with respect to the initial system by an angle ϕ.

The balance of moments of momentum yields

$$\curvearrowright \quad (\tau_{yx} d_x)\frac{1}{2}\, dy - (\tau_{xy}\, dy)\frac{1}{2}\, dx = 0$$

i.e., the relation of Cauchy

$$\tau_{yx} = \tau_{xy} \tag{10}$$

We obtain the expressions of the projections of stress in the ξ, η directions of two prisms (above) if it is taken into account that /159

$$\longmapsto \quad dx = d\xi \cos\varphi \;, \quad dy = d\xi \sin\varphi$$ and then:

$$\begin{aligned}
\sigma_\xi &= \sigma_x \cos^2\varphi + \sigma_y \sin^2\varphi - 2\tau_{xy} \sin\varphi \cos\varphi \\
\sigma_\eta &= \sigma_x \sin^2\varphi + \sigma_y \cos^2\varphi - 2\tau_{xy} \sin\varphi \cos\varphi \\
\tau_{\xi\eta} &= (\sigma_y - \sigma_x) \sin\varphi \cos\varphi + \tau_{yx} (\cos^2\varphi - \sin^2\varphi)
\end{aligned} \tag{11}$$

or if we use the angle 2ϕ, we have:

$$\begin{aligned}
\sigma_\xi &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - (-\tau_{xy}) \sin 2\varphi \\
\sigma_\eta &= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + (-\tau_{xy}) \sin 2\varphi \\
-\tau_{\xi\eta} &= \qquad \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\varphi + (-\tau_{xy}) \sin 2\varphi
\end{aligned} \tag{12}$$

Equations (11) and (12) give the stresses with respect to the system of axes ξ, η in terms of the stresses with respect to the system (x,y). Therefore, the symmetric matrix defines completely the stress state at a point xy $\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix}$.

Observe that for the definition of the stress state 2^2 elements are required (two dimensions). On the other hand, the matrix $\begin{pmatrix} \sigma_\xi & \sigma_{\xi\eta} \\ \tau_{\eta\xi} & \sigma_\eta \end{pmatrix}$ is obtained from Eqs. (12), which are identical to the equations of transformation of the inertia tensor for rotation. Therefore, based on the given definition, the above matrix represents a symmetric tensor of second rank of a certain quantity at the point (x,y). Since this quantity describes the stress state at the point under consideration, it is called the stress tensor (i.e. Eqs. (12) give the transformation of the inertia tensors. On the other hand, the one matrix is derived from the other by a similar transformation. Therefore, as was shown in the example, it is a tensor.)

Correspondingly, we have the fact that the nine entry symmetric matrix

$$\begin{pmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix}$$

defines completely the stress state in space, and is called the stress tensor.

If we choose as directions of the axes the three principal directions, the stress tensor is diagonalized: /160

$$\begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_x & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}$$

Closing the introduction to the theory of elasticity, we must point out that the main advantage of the tensors is that, knowing the stress state for one cross section, we can find the stress state for any other cross section (it can be proved).

The deformation can also be represented by tensors. If we consider the strains ε_x, ε_y of the sides dx, dy of an orthogonal triangle and the change of the right angle γ_{xy} of this triangle, the nine entry symmetric matrix represents the strain tensor. This can be proven, as we have done before, since it obeys the same transformation laws.

$$\begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 1/2\,\gamma_{yz} \\ 1/2\,\gamma_{zx} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{pmatrix}$$

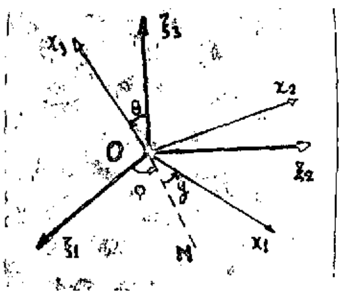
We have given the definitions of stress and strain tensors, considering a material point of a body. Macroscopically, we consider that the Earth -- at least in the newer theory -- deforms elastically. By this we mean that if the external cause of deformation is eliminated, the Earth regains its initial (before deformation) shape. This helps us to find the inertia tensors, assuming that the principle of superposition holds. Thus we can obtain for an elastic Earth the inertia tensor as the sum of a constant inertia tensor I_0 of the undeformed Earth and of I_ω accounting for the deformation. In this way we can find the change of the inertia tensor due to deformation (see newer theory).

8.1. Introduction

Before we try to develop the mathematical foundation of polar motion, we will define certain concepts which are necessary for the understanding of the following.

1. We say that a system of material points constitutes a rigid body when the relative distances among all the material points remain constant during motion. Considering the motion of a rigid body, the projections of the velocities of two arbitrary points on the straight line passing through them are equal. This is a characteristic property of the motion of a rigid body.

2. In order to determine the position of a rigid body which moves freely around a fixed point O (in general), we can make use of three angles ϕ, ψ, θ, which are called Euler angles. If we consider a "fixed" reference coordinate system $O\xi_1\xi_2\xi_3$ and a system $Ox_1x_2x_3$ attached on the rigid body, and if ON is the intersection of the plane Ox_1x_2 with the fixed plane $O\xi_1\xi_2$ we define the angles as:



$$\phi = \xi_1 \hat{O} N, \quad \psi = N\hat{O}x_1; \quad \theta = \xi_3\hat{O}x_3.$$

It is obvious that the position of the rigid body is completely determined if the three Euler angles ϕ, ψ, θ are given.

3. The inertia system is the coordinate system with respect to which the laws of Newton hold. The space and time of an inertia system are homogeneous and isotropic.

4. We consider necessary to state the theorem of balance of angular momentum. Besides its general application to the whole /162 theory, which we will use for the special case of zero external forces, it constitutes the basis of the forced polar motion due to action of forces on the surface of the Earth.

Consider $\overline{G}$ = vector of angular momentum
$\overline{\Omega}$ = angular velocity
$\overline{L}$ = external torque

We have $\overline{G} = \overline{r} \wedge \overline{p}$ where $\overline{p} = m\overline{v}$, vector of linear momentum

$$\rightsquigarrow \frac{d\bar{G}}{dt} = \frac{d\bar{r}}{dt} \wedge \bar{p} + \bar{r} \wedge \frac{d\bar{p}}{dt} = \bar{v} \wedge \bar{p} + \bar{r} \wedge \frac{d\bar{p}}{dt} \Rightarrow$$

but

$$\Rightarrow \left\{ \begin{array}{l} \bar{v} /\!/ \bar{p} \rightarrow \bar{v} \wedge \bar{p} = 0 \\ \frac{d\bar{G}}{dt} = \bar{r} \wedge \frac{d\bar{p}}{dt} = \bar{r} \wedge \bar{F} = \bar{L} \end{array} \right\} \rightsquigarrow \frac{d\bar{G}}{dt} = \bar{L}$$

i.e. the rate of change of angular momentum equals the applied torque I.

(Note: Newton's Second Law: $\Sigma\bar{F} = d\bar{p}/dt$).

5. Consider a fixed reference system $O\xi_i$ with unit vectors ε_i and a system Ox_i with unit vectors e_i, moving with an angular velocity ω. Consider a variable vector

$$\bar{G} \longrightarrow (1)\ \bar{G} = G_{\xi_1}\bar{\varepsilon}_1 + G_{\xi_2}\bar{\varepsilon}_2 + G_{\xi_3}\bar{\varepsilon}_3 = G_{x_1}\bar{e}_1 + G_{x_2}\bar{e}_2 + G_{x_3}\bar{e}_3$$

We call the "total" derivative the derivative of $\bar{G}$ with respect to the system $O\xi_i$

$$\rightsquigarrow \frac{d_a\bar{G}}{dt} = \dot{G}_{\xi_1}\bar{\varepsilon}_1 + \dot{G}_{\xi_2}\bar{\varepsilon}_2 + \dot{G}_{\xi_3}\varepsilon_3$$

We call the "relative" derivative the derivative of $\bar{G}$ with respect to the system Ox_i, i.e.

$$\rightsquigarrow \frac{d_6\bar{G}}{dt} = \dot{G}_{x_1}\bar{e}_1 + \dot{G}_{x_2}\bar{e}_2 + \dot{G}_{x_3}\bar{e}_3$$

We take the derivative of $\bar{G}$:

$$\dot{G}_{\xi_1}\bar{\varepsilon}_1 + \dot{G}_{\xi_2}\bar{\varepsilon}_2 + \dot{G}_{\xi_3}\bar{\varepsilon}_3 = \dot{G}_{x_1}\bar{e}_1 + \dot{G}_{x_2}\bar{e}_2 + \dot{G}_{x_3}\bar{e}_3 + G_{x_1}\dot{\bar{e}}_1 + G_{x_2}\dot{\bar{e}}_2 + G_{x_3}\dot{\bar{e}}_3$$

$$\rightsquigarrow \boxed{\frac{d_a\bar{G}}{dt} = \frac{d_6\bar{G}}{dt} + \bar{\omega} \wedge \bar{G}}$$

then

$$(\dot{\bar{e}}_1 = \bar{\omega} \wedge \bar{e}_1,\quad \dot{\bar{e}}_2 = \bar{\omega} \wedge \bar{e}_2,\quad \dot{\bar{e}}_3 = \bar{\omega} \wedge \bar{e}_3)$$

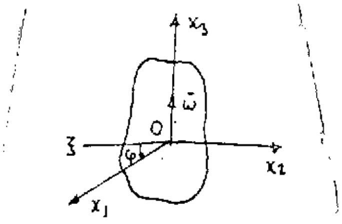
$$\rightsquigarrow G_{x_1}\bar{\omega} \wedge \bar{e}_1 + G_{x_2}\bar{\omega} \wedge e_2 + G_{x_3}\bar{\omega} \wedge \bar{e}_3$$

$$\rightsquigarrow \bar{\omega} \wedge G_{x_1}e_1 + \omega \wedge G_{x_2}e_2 + \omega \wedge G_{x_3}\bar{e}_3$$

$$\rightsquigarrow \bar{\omega} \wedge \bar{G}$$

It gives the relation between total and relative derivative of a vector with respect to a system moving with angular velocity $\bar{\omega}$. /163

8.2. Rotation of a Rigid Body around a Fixed Axis



Consider a rigid body which rotates around a fixed axis, passing through the point O, with angular velocity $\bar{\omega}$. if $\bar{e}_3$ is the unit vector on the axis of rotation OZ, then it is

$$\bar{\omega} = \omega\bar{e}_3$$

Consider a coordinate system $Ox_1x_2x_3$ where $OZ \equiv Ox_3$ and a known fixed direction $O\xi$. Obviously the plane Ox_1x_2 is fixed, and if $\phi = \xi\hat{O}x$, then we will have the relation $\phi = \omega$ (8.2.1). The following formula is given:

$$\bar{G} = (J_{11}\omega_1 + J_{12}\omega_2 + J_{13}\omega_3)\bar{e}_1 + (J_{21}\omega_1 + J_{22}\omega_2 + J_{23}\omega_3)\bar{e}_2 + (J_{31}\omega_1 + J_{32}\omega_2 + J_{33}\omega_3)\bar{e}_3 \tag{8.2.2}$$

Substituting in the formula above $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$, $J_{33} = J$

$$\Rightarrow \bar{G} = \omega\,(J_{13}\bar{e}_1 + J_{23}\bar{e}_2 + J\bar{e}_3) \tag{8.2.3}$$

The theorem of balance of angular momentum yields:

$$\frac{d\bar{G}}{dt} = \dot{\bar{G}} = \bar{L} \tag{8.2.4}$$

From (8.2.3) we have:

$$\dot{\bar{G}} = \dot{\omega}(J_{13}\bar{e}_1 + J_{23}\bar{e}_2 + J\bar{e}_3) + \omega\,(J_{13}\dot{\bar{e}}_1 + J_{23}\dot{\bar{e}}_2) = \bar{L}, \tag{8.2.5}$$

I: torque of external forces.

The unit vectors and the angular velocity are related through relations of the form

$$\dot{\bar{e}}_1 = \bar{\omega}\wedge\bar{e}_1,\quad \dot{\bar{e}}_2 = \bar{\omega}\wedge\bar{e}_2,\quad \dot{\bar{e}}_3 = \bar{\omega}\wedge\bar{e}_3 \Rightarrow$$

$$\Rightarrow \dot{\bar{e}}_1 = \omega\bar{e}_2,\quad \dot{\bar{e}}_2 = -\omega\bar{e}_1$$

Then (8.2.5) yields finally:

$$\dot{\bar{G}} = (J_{13}\dot{\omega} - J_{23}\omega^2)\bar{e}_1 + (J_{23}\dot{\omega} + J_{13}\omega^2)\bar{e}_2 + J\dot{\omega}\bar{e}_3 = \bar{L} \tag{8.2.6}$$

If L_1, L_2, L_3 are the projections of the torque $\bar{L}$ with respect to the axes Ox_1, Ox_2, Ox_3 correspondingly, from (8.2.6)

$$\left.\begin{aligned} J_{13}\dot{\omega} - J_{23}\omega^2 &= L_1 \\ J_{23}\dot{\omega} + J_{13}\omega^2 &= L_2 \\ J\dot{\omega} &= L_3 \end{aligned}\right| \tag{8.2.7}$$

Considering Eq. (8.2.1), $\dot{\phi} = \omega$, the third part of (8.2.7) becomes /164

$$J\dot{\phi} = L_3 \tag{8.2.8}$$

Assume that the Earth has the shape of an ellipsoid with two equal axes. We take as a coordinate system $Ox_1x_2x_3$ the system of the principal axes of the ellipsoid. The products of inertia are zero with respect to these axes. Thus the system of principal axes coincides with the system of principal directions (principal axes of inertia).

We consider that the axis of rotation of the Earth (instantaneous) coincides with the principal axis of the ellipsoid and we assume that the external forces are zero. As external forces we consider the applied forces and the reactions of the axis. If the applied forces are zero, then the reactions pass through the axis Ox_3 and therefore have zero moment with respect to the axis Ox_3.

Therefore, $L_3 = 0$. From the equation $J\ddot{\phi}\omega = L_3 \rightarrow \omega$ = constant. Also, since the products of inertia are all zero, the remaining quantities of (8.2.7) vanish, i.e.

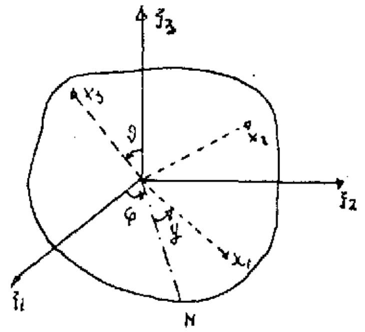
$$L_1 = L_2 = 0$$

So we consider the following:

If the instantaneous axis of the Earth coincides with the principal axis of the ellipsoid and if the external couple acting on the Earth is zero, then the angular velocity ω will remain constant and the axis will keep its initial position ($d\bar{G}/dt = 0 \rightarrow \bar{G}$ = constant).

8.3. Motion of a Rigid Body Around a Fixed Point with Moment of External Forces Different from Zero

Consider a rigid body which can freely rotate around a fixed point O. Consider also a coordinate system $Ox_1x_2x_3$ rigidly /165 attached to the body and $O\xi_1\xi_2\xi_3$ an inertial system which we will use as a reference system for the motion of the body. Since we have three degrees of freedom, we need three parameters to determine the position of the body with respect to the axes $O\xi_1\xi_2\xi_3$. We choose the three Euler angles ϕ, ψ, θ. To study the motion of the rigid body, we will use the theorem of balance of angular momentum, $d\overline{G}/dt = \overline{L}$, where $\overline{G}$ is the angular momentum with respect to the fixed point O and $\overline{L}$ is the torque of all the external forces with respect to O. The derivatives are taken with respect to the inertial system $O\xi_1\xi_2\xi_3$



We assume that the system $Ox_1x_2x_3$ coincides with the principal axes of the ellipsoid of inertia (coinciding with the principal axes of inertia-principal directions). In this case, the vector of angular momentum takes the form

$$\overline{G} = \lambda_1 \omega_1 \overline{e}_1 + \lambda_2 \omega_2 \overline{e}_2 + \lambda_3 \omega_3 \overline{e}_3 \tag{8.3.1}$$

The last relation is derived from (8.2.2) if we set $J_{i\lambda} = 0$, $J_{ii} = J_i$ $(i \neq \lambda)$, where the ω_1, ω_2, ω_3 are the projections of the instantaneous angular velocity $\overline{\omega}$ of the rigid body on axes Ox_1, Ox_2, Ox_3 correspondingly and $\overline{e}_i$ are the unit vectors on the principal axes.

As we have mentioned, the derivative $d\overline{G}/dt$ is taken with respect to the inertial system $O\xi_1\xi_2\xi_3$. It was shown that the derivative $d\overline{G}/dt$ is related to the "relative" derivative of $\overline{G}$ with respect to Oxyz, by a relation of the form:

$$\frac{d\overline{G}}{dt} = \frac{d_c\overline{G}}{dt} + \overline{\omega} \wedge \overline{G} \tag{8.3.2}$$

where

$$\frac{d_c\overline{G}}{dt} = J_1 \dot{\omega}_1 \overline{e}_1 + J_2 \dot{\omega}_2 \overline{e}_2 + J_3 \dot{\omega}_3 \overline{e}_3 \tag{8.3.3}$$

If we also take into account that we have

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \omega_3 \bar{e}_3 \qquad (8.3.4)$$

and that /166

$$\frac{d\bar{G}}{dt} = \bar{L} \Rightarrow \frac{d_6 \bar{G}}{dt} + \bar{\omega} \wedge \bar{G} = \bar{L} \qquad (8.3.5)$$

we obtain from (8.3.5) the system:

$$\left.\begin{aligned} J_1 \dot{\omega}_1 - (J_2 - J_3)\,\omega_2 \omega_3 &= L_1 \\ J_2 \dot{\omega}_2 - (J_3 - J_1)\,\omega_3 \omega_1 &= L_2 \\ J_3 \dot{\omega}_3 - (J_1 - J_2)\,\omega_1 \omega_2 &= L_3 \end{aligned}\right\} \qquad (8.3.6)$$

L_i are the projections of the moments of external forces with respect to the point O, on the axes Ox_i, correspondingly. Equations (8.3.6) are called the Euler equations.

The solution of the system (8.3.6) will give the motion of the instantaneous axis of rotation with respect to the moving body. If we want to find the motion of the instantaneous axis of rotation with respect to the fixed axes, we must express the components ω_i of the angular velocity in terms of the Euler angles.

8.4. Motion of a Rigid Body Around a Fixed Point, with Moment of External Forces Equal to Zero /167

We will consider the special case of zero external forces. Then the moments also of the external forces with respect to point O will be zero. Therefore, the Euler equations become:

$$\left.\begin{aligned} J_1 \dot{\omega}_1 - (J_2 - J_3)\,\omega_2 \omega_3 &= 0 \\ J_2 \dot{\omega}_2 - (J_3 - J_1)\,\omega_3 \omega_1 &= 0 \\ J_3 \dot{\omega}_3 - (J_1 - J_2)\,\omega_1 \omega_2 &= 0 \end{aligned}\right\} \qquad (8.4.1)$$

This is the case where the fixed point O coincides with the center of mass of the body and the external forces are only the weight of the body.

If we multiply (8.4.1) by ω_1, ω_2, ω_3, respectively, and we add

$$\rightsquigarrow J_1\omega_1\dot{\omega}_1 + J_2\omega_2\dot{\omega}_2 + J_3\omega_3\dot{\omega}_3 = \frac{1}{2}\frac{d}{dt}\left(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2\right) = 0 \Longrightarrow$$

$$\Longrightarrow T = \frac{1}{2}\left(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2\right) = \text{const} \tag{8.4.2}$$

Also if we multiply (8.4.1) by $J_1\omega_1$, $J_2\omega_2$, $J_3\omega_3$, respectively, and we add

$$\rightsquigarrow J_1^2\omega_1\dot{\omega}_1 + J_2^2\omega_2\dot{\omega}_2 + J_3^2\omega_3\dot{\omega}_3 = \frac{1}{2}\frac{d}{dt}\left(J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2\right) = 0$$

$$\Longrightarrow L^2 = J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2 = \text{const} \tag{8.4.3}$$

Equations (8.4.2) show that the kinetic energy of the body remains constant and Eq. (8.4.3) shows that the norm of the angular momentum with respect to the fixed point is constant. The solution of the system (8.4.1) gives the motion of the instantaneous axis of rotation and the instantaneous angular velocity with respect to the body. In order to determine the position of the body in space, we must have the Euler angles as functions of time. Since $\bar{L} = 0$ (angular momentum constant $d\bar{G}/dt = 0$), we choose the fixed reference coordinate system $O\xi_1\xi_2\xi_3$ in such a way as to have the axis $O\xi_3$ coinciding with the direction of the vector of angular momentum. Then: /168

$$\text{System } O\xi_1\xi_2\xi_3 \longrightarrow \bar{G} = G\bar{\epsilon}_3$$
$$\text{System } Ox_j \longrightarrow \bar{G} = J_1\omega_1\bar{e}_1 + J_2\omega_2\bar{e}_2 + J_3\omega_3\bar{e}_3 \tag{8.4.4}$$

But having G coinciding with $O\xi_3$, we can find the projections on Ox_1 using the Euler angles

$$\rightsquigarrow G_1 = G\sin\varphi\sin\vartheta, \quad G_2 = G\cos\varphi\sin\vartheta, \quad G_3 = G\cos\vartheta \tag{8.4.5}$$

From (8.4.4) and (8.4.5), we have:

$$J_1\omega_1 = G\sin\psi \,.\sin\vartheta$$
$$J_2\omega_2 = G\cos\psi \,.\sin\vartheta \qquad (8.4.6)$$
$$J_3\omega_3 = G\cos\vartheta$$

and we can express the angles θ, ψ, ϕ in terms of $\omega_1(t)$

$$\Rightarrow \cos\vartheta = \frac{J_3\omega_3}{G}, \quad \tan\psi = \frac{J_1\omega_1}{J_2\omega_2}, \quad \dot{\varphi} = \frac{\omega_3 - \dot{\psi}}{\cos\vartheta} \qquad (8.4.7)$$

8.5 Analytical Solution for the Free Motion of a Body with an Axis of Symmetry

In the above, the Euler equations were given for the general case of a motion and then the case of zero external forces was considered. Considering now the Earth, we will first assume the existence of an axis of symmetry because of our earlier assumption about the shape of an ellipsoid.

In this case, we have the same moments of inertia with respect to two principal axes $J_1 = J_2$, and if $J_3 = J$, the equations (8.4.1) take the form:

$$J_1\dot{\omega}_1 = (J_1 - J)\omega_2\omega_3$$
$$J_1\dot{\omega}_2 = -(J_1 - J)\omega_1\omega_3 \qquad (8.5.1)$$
$$J\dot{\omega}_3 = 0$$

$J\dot{\omega}_3 = 0 \rightarrow \omega_3$ = constant. Taking the derivative of the first of (8.5.3) and using the second equation: /169

$$\ddot{\omega}_1 + \left[\left(\frac{J_1 - J}{J_1}\right)^2 \omega_3^2\right]\omega_1 = 0 \qquad (8.5.2)$$

We will solve it:

$$\longrightarrow \left\{ \begin{array}{l} \dfrac{J_1 - J}{J_1}\omega_3 = B \\ \ddot{\omega}_1 + B^2\omega_1 = 0 \end{array} \right\}$$

The corresponding homogeneity is $D^2 + B^2 = 0 \rightarrow D = \pm iB$. So the solution will have the form

$$\rightsquigarrow \omega_1 = \Gamma e^{-iBt} + A e^{iBt} \Rightarrow$$

$$\Rightarrow \underline{\omega_1 = \Gamma \cos Bt + A \sin Bt} \tag{8.5.3}$$

Substituting in the first part of (8.5.1) and performing the calculations, we obtain

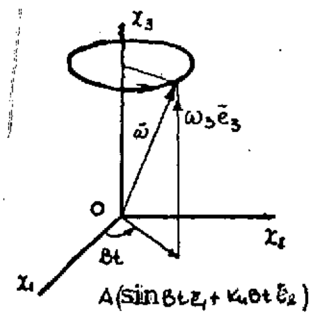
$$\Rightarrow \dot{\omega}_1 = B\dot{\omega}_2 \Rightarrow \underline{\omega_2 = -\Gamma \sin Bt + A \cos Bt}$$

A, Γ are constants, defined by the initial conditions.

We finally find, for the position of the vector of the instantaneous angular velocity with respect to a moving body:

$$\rightsquigarrow \bar{\omega} = A(\sin Bt\, \bar{e}_1 + \cos Bt\, \bar{e}_2) + \omega_3 \bar{e}_3 \tag{8.5.4}$$

(because ω_i = projections of $\bar{\omega}$ with respect to Ox_i). Since $\omega_3 = \cos\theta$, it is $|\bar{\omega}|$ = constant. We observe that the vector $(\sin Bt\bar{e}_1 + \cos Bt\bar{e}_2)$ rotates with respect to the body with constant angular velocity B. That is, the position of the instantaneous axis of rotation is not fixed with respect to the moving body, but it rotates around the axis of symmetry of the body with angular velocity B. Therefore, it completes a full rotation in time



$$t = \frac{2\pi}{B} = \frac{2\pi J_1}{(J_1 - J_3)\omega_3} \tag{8.5.5}$$

Having found the instantaneous angular velocity with respect to the body, the motion of the body in space is determined with the help of Eq. (8.4.7), i.e. of:

$$\cos\vartheta = \frac{J_1\omega_3}{G}, \quad \tan\psi = \frac{J_1\dot{\omega}_1}{J_2\omega_2}, \quad \dot{\varphi} = \frac{\omega_3 - \dot{\psi}}{\cos\vartheta}$$

ω_3 = constant ⇉ θ = constant; also $\tan\psi = \frac{\omega_1}{\omega_2} = \tan Bt \Rightarrow \psi = Bt$ /170

Therefore, $\dot\phi$ = constant ⇉ $\phi = \frac{G}{J_1}t$, G = modulus.

So we have

$$\theta = \text{constant}, \quad \psi = Bt, \quad \varphi = \frac{G}{J_1}t \tag{8.5.6}$$

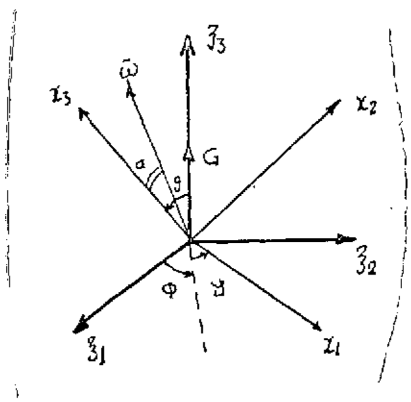
Therefore, the rigid body moves in such a way as the axis Ox_3 has a constant angle with $O\xi_3$ ($\theta = \cos\theta$) and the angles ϕ, ψ are linear functions of time. The instantaneous angular velocity in terms of the Euler angles is given by the relation

$$\bar\omega = \dot\vartheta \bar n + \dot\varphi \bar\epsilon_3 + \dot\psi \bar e_3 \tag{8.5.7}$$

where $\bar n$ = unit vector of the axis ON, if $\bar\omega$ is the rotation vector with respect to the axes ON, $O\xi_3$, Ox_3. Taking account of the previously given relations,

$$\bar\omega = \dot\varphi \bar\epsilon_3 + \dot\psi \bar e_3 = \frac{G}{J_1}\bar\varepsilon_3 + B\bar e_3 \tag{8.5.8}$$

Therefore, the vector $\bar\omega$ lies on the plane of the axes Ox_3 and $O\xi_3$, and because the plane of the axes $O\xi_3$, Ox_3 rotates with angular velocity $\dot\phi = \frac{G}{J_1}$, the vector $\bar\omega$ ($\bar\omega = \dot\phi$) will rotate with respect to the fixed system $O\xi_1$ around the $O\xi_3$ axis, with the same angular velocity, and will describe a conical surface. With respect to the observer on the body, the vector $\bar\omega$ describes also a conical surface with axis Ox_3 and constant angular velocity B (8.5.4). The two cones are tangent along the instantaneous axis of rotation rolling on each other. This motion of the body is called regular precession. The angle α between the vector of the instantaneous angular velocity $\bar\omega$ and the axis Ox_3 will be:



$$\tan\alpha = \frac{|\bar\omega \wedge \bar e_3|}{\bar\omega \cdot \bar e_3} \tag{8.5.9}$$

Also $\left\{ B = \frac{J_1 - J}{J_1}\omega_3 , \quad \cos\vartheta = \frac{J}{G}\omega_3 \right\} \Rightarrow \quad B = G\,\frac{J_1 - J}{J_1 J}\cos\vartheta$

$$(8.5.10)$$

and then Eq. (8.5.8) becomes

$$\longrightarrow \bar{\omega} = \frac{G}{J_1}\left(\bar{\varepsilon}_3 + \frac{J_1 - J}{J}\cos\vartheta\ \bar{e}_3\right) \qquad (8.5.11)$$

Substituting (8.5.11) in (8.5.9), and taking account of /171

$$\bar{\varepsilon}_3\,\bar{e}_3 = \cos\vartheta , \quad \left|\bar{\varepsilon}_3 \wedge \bar{e}_3\right| = \sin\vartheta \longrightarrow \tan\alpha = \frac{J}{J_1}\tan\vartheta \qquad (8.5.12)$$

From the relation (8.5.12), we have that if $J < J_1 \to \alpha < \theta$ (Fig. 1), $J > J_1 \to \alpha > \theta$ (Fig. 2). The above are true for the case of the free motion of a rigid body with zero external forces, or for the case of the motion of a rigid body around a fixed point when the moments of the external forces with respect to the point are zero. But all the above are also applied to the case of the free motion of a rigid body in space, when the applied forces have zero moment with respect to the center of the body, because again system (8.4.1) holds. This happens, for example, when the applied forces are derived from a homogeneous gravity field and the body has spherical symmetry with respect to its mass distribution.

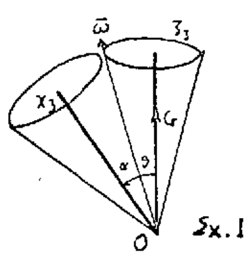


Fig. 1.

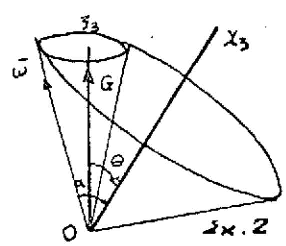


Fig. 2.

The case of the motion of the Earth is an example of the above, because we have imposed the following conditions:

1. System Ox_i = system of principal axes of inertia (principal directions).

2. Vector of angular momentum $\bar{G} = G\bar{\varepsilon}_3$, i.e. parallel to $O\xi_3$.

3. Existence of an axis of symmetry.

4. The axis Ox_3 is tilted.

5. Absence of external torques and forces derived only from a homogeneous gravity field.

Therefore, for the case of the Earth, considered as an ellipsoid by revolution with $J_1 = J_2 < J$, moment of inertia with respect to the principal axis of symmetry given by J and having the axis Ox_3 tilted, it was proved that this axis will follow a precessional motion with angular velocity B with respect to the moving coordinate system Ox_i. And since $-J_1/(J_1 - J) \simeq 300$ and $\omega_3 \simeq \omega$, where ω = angular velocity of rotation of the Earth, we will have that the instantaneous axis of rotation of the Earth will not be fixed with respect to the Earth, but it will rotate around its axis of symmetry completing a full revolution in $\simeq 300$ days (304). That is, the vector of the instantaneous angular velocity of the Earth sweeps a conical surface with respect to the Earth with axis the axis of symmetry of the Earth. The amplitude of this rotational motion is very small, on the order of a few meters. /172

Summarizing, in the absence of external forces, if the axis of inertia of the Earth coincides with the axis of rotation, the Earth will continue to rotate around the principal axis of inertia; otherwise, it will describe a conical surface with its center the center of mass of the Earth and axis Ox_3 the principal axis of inertia. But in reality, the period of motion of the instantaneous axis is different, because the Earth can not be considered as a rigid body, and because there are external forces acting on it.

8.6. Theory of Forced Motion

In this chapter we will examine, for a rigid body, the results of the action of external forces which appear because of physical phenomena.

Given $\bar{G}$ = vector of angular momentum,
$\bar{\Omega}$ = angular velocity,
$\bar{L}$ = externally acting torque,

we will have: $\bar{G} = \bar{r} \wedge \bar{p}$ where $\bar{p} = m\bar{v}$, vector of linear momentum /173

$$\rightsquigarrow \frac{d\bar{G}}{dt} = \frac{d\bar{r}}{dt} \wedge \bar{p} + \bar{r} \wedge \frac{d\bar{p}}{dt} = \bar{v} \wedge \bar{p} + \bar{r} \wedge \frac{d\bar{p}}{dt}, \text{ but } \bar{v} // \bar{p} \Rightarrow \bar{v} \wedge \bar{p} = 0$$

(by definition, $\Sigma\bar{F} = \frac{d\bar{p}}{dt}$ B', Newton's law)

$$\rightsquigarrow \frac{d\bar{G}}{dt} = \bar{r} \wedge \frac{d\bar{p}}{dt} = \bar{r} \wedge \bar{F} = \bar{L}$$

$$\frac{d\bar{G}}{dt} = \bar{L}$$

i.e. the rate of change of angular momentum is equal to the applied torque.

$$\bar{G} = G\bar{k} \Rightarrow \bar{L} = \frac{dG}{dt}\bar{k} + G\frac{d\bar{k}}{dt}$$

This means that the torque applied on the body changes the magnitude of the angular momentum in the direction of the torque, and also it changes the course of the angular momentum. Obviously, the second term is significant only when the direction of the torque is different from the initial direction of the angular velocity. We consider a torque applied on a body which can rotate around an axis able to move (rotate). Any torque can be analyzed in two components, one in the direction of the axis and one normal to it. The first component, having the direction of the rotational axis, coincides with respect to direction with the angular momentum. The result is the change in magnitude of the angular momentum.

Of greater significance is the case of the normally applied component. We will analyze this case in detail, because exactly this changes the direction of the axis, and it is the main reason for it. It is therefore enough to consider only the application of a vector of normal torque, since the parallel vector will only change the magnitude of the angular momentum. For definiteness, consider a rigid body s h a p e d by revolution which can freely /174
rotate (top).

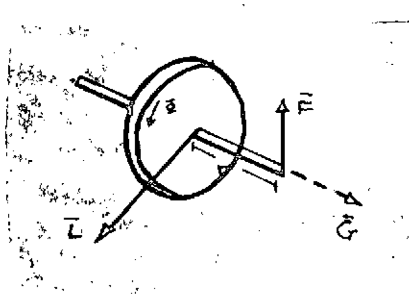


Fig. 1

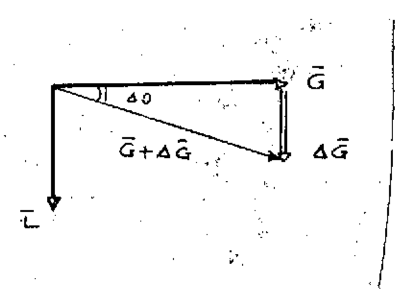


Fig. 2.

Assume $\bar{\Omega}$ is the angular velocity. If external forces are not applied, the body will move with a constant angular velocity Ω and constant angular momentum. Consider that on the one tip of the axis of the disc a perpendicular force is acting. It will exercise torque $\bar{L} = \bar{r} \wedge \bar{F}$ where $\bar{r}$ is the distance of the point of application from the center of mass. (We consider the center of mass as the origin.)

Due to this torqe, the vector $\bar{G}$ will change (as is already shown). This change is $\bar{L} = d\bar{G}/dt$.

Considering $\bar{L}$ constant, we obtain $\Delta\bar{G} = \bar{L}\Delta t$.

But the change in the vector of angular momentum will be in the direction of the torque; therefore, it will be perpendicular to the initial vector (see Eq. (2)). The final result is given from the vector sum $\bar{G} + \Delta\bar{G}$, and it is

$$\tan\Delta\theta = \frac{\Delta G}{G} \simeq \Delta\theta \text{ (rad)}$$

i.e. the initial vector rotated an amount $\Delta\theta$. Thus the application of the torque on the disc has as a result the change in direction of the angular momentum.

Assume that the acting force $\bar{F}$ is constant and continuously perpendicular. Therefore, torque $\bar{L}$ will also be constant and, moreover, it will lie on the horizontal plane. Therefore the rotational axis will rotate on the horizontal plane with constant velocity, because $\bar{L}$ is constant. We therefore observe that the external torque applied on the axis of rotation of a rotating disc causes continuous rotation of this axis around an axis perpendicular to the initial one. /175

The velocity of rotation $\bar{\Omega}$ will be given by $\Omega = \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$. But we have shown that $\Delta\bar{G} = \bar{L}\Delta t$ and $\tan\theta \simeq \Delta\theta^r = \Delta G/G \Rightarrow \Omega = L/G$. That is, the magnitude of triangular velocity of the rotation of the axis is equal to the ratio of the magnitude of the external torque over the magnitude of the initial angular momentum (norm of "resistance"). The angular velocity $\bar{\Omega}$ is called velocity of precession of the axis of rotation, and it is perpendicular to $\bar{G}$, i.e. $\bar{L} = \bar{\Omega} \wedge \bar{G}$. Also taking account of the increase in angular momentum because of the parallel component, the total effect of the action of the torque on the body will be

$$\boxed{\bar{L} = \dot{G}\bar{K} + \bar{G} \wedge \bar{\Omega}} \quad \text{(differentiation } \bar{G} = C\bar{K}\text{)}$$

The phenomenon described above is fundamental for the theory of forced nutation, i.e. of the nutation of the pole due to externally applied forces (meteorological tide).

From the previously given relation

$$\bar{L} = \frac{d\bar{G}}{dt}\bar{K} + G\frac{d\bar{K}}{dt} \Rightarrow \left| G\frac{d\bar{K}}{dt} = \bar{G} \wedge \bar{\Omega} \right|$$

i.e. the result of the externally acting torque on the velocity of precession is proportional to its value.

9. A NEWER MATHEMATICAL THEORY OF THE MOTION OF THE EARTH

9.1. Introduction

The classical theory of rotational motion, based on the assumption that the Earth is a rigid body, was completed toward the end of the last century. Around 1882, Oppolzer obtained the formula of precession and nutation in a form which has been used /176 up to the present. Therefore, reconsiderations of the theory appeared only under the form of improving the accuracy of the values of certain constants entering the formulas. In this theory, certain important inaccuracies and omissions were not discovered. It was also not noted that the results were different in some cases from the observations made. It was already recognized that the Earth is not a rigid sphere. But no serious attempt was made to reconsider the theory on the basis of new assumptions about the mechanical properties of the Earth.

This point of view changed when the 14-month change in latitude was discovered. With harmonic analysis of the results of the latitude observations, Chandler tried to make an accurate evaluation of the Euler theory of polar motion. In any case, his work played an important role because the results of the observations were in disagreement with the theory. These had shown that it was necessary to reconsider the main hypothesis of the rigidity of the Earth. At the same time, scientists obtained a criterion for the validity of another assumption about the properties of the Earth which would yield a theoretically predicted value of free nutation equal to the observed one, i.e. of 14 months.

The above criterion was used in order to test the assumption of the liquid core of the Earth, and finding that this hypthesis leads to a reduction of the free period, we conclude that this assumption was wrong. Newcomb considered the elastic deformation of the Earth during its rotation and showed, from simple hypo- /177 theses, that such a deformation indeed increases the free period. This was proved in the work of Schweydar, i.e that this increase in period was the only indication of the elasticity which would be discovered in analyzing the astronomical observations. In other cases, the Earth could practically be considered as rigid. Though the formula for precession and nutation was initially obtained under the hypothesis that the Earth is rigid, we could as well apply the same procedure when we consider the Earth as an elastic body.

New considerations would be necessary if some discrepancies existed between theory and observations. Then the theoretical research would have a specific goal, and it would not seem to be simply a search for solutions of abstract problems of mechanics. Therefore, it is desirable to have a complementary comparison between the predictions of a theory which considers the Earth ideally elastic and the astronomical observations.

The present work is an attempt to make such a comparison. In Chapter 1 the theory is given, using vector and tensor analysis. To justify this, an abandoning of the classical procedure in deriving the equations for precession and nutation is enough to show that the tensor analysis method is generally accepted in the mechanics of rigid bodies.

In only two works is the effect of an externally applied force on the rotation of an elastic Earth considered. These works are by Schwedar and Sekiguchi. The work of Sekiguchi was /178 based on the results of a previous work of his, in which Woolard showed that there is an error in the principles. In what follows, use will be made of the work of Schweydar, as a way to check on the results that we obtain using other methods. Note that we should decide which of the theoretical results are more important for comparison with the observations. We first consider the constant of nutation N. In a compact presentation of this matter, Idelson explained the importance of the following paragraph. "In 1930 Sitter took as the value of N

$$N = 9''.2075 \pm 0''.0055$$

In any case, by his method the value N = 9".2181 was obtained. This value was chosen as a consequence of the theory of a rigid Earth with the value of precession and the determination by Hink of the precession of the sun and the ascension of the moon, i.e. it is a theoretical value. Later, they incorporated a small change. Jakson considered this discrepancy as 'one of the most important discrepancies of the constants of the solar system,' and stated that the relative results in the precession could not compensate for the results of the relativity among the constant relationships. Bronwer noted that de Sitter unsuccessfully searched for the solution of the problem during his last year. And, even more, "from the previous summary it can be seen that the matter concerning the constant of nutation has not reached a final conclusion." The analysis of the enormous amount of material given by I.L.S. gives a value for N smaller than the one /179 theoretically predicted.

Recently Jeffrey tried to explain the discrepancy by considering a fluid core in the Earth, and as far as we know, he is still continuing his research. The results of Przybyllok, which

Idelson considered to be very important, were taken from information by I.L.S. during 1900-1925. We now have much more information. The results of the I.L.S. published up to the present cover two cycles of nutation. It seems to us important to determine first the constant of nutation, and then to find whether the new result for N approaches the theoretical value or deviates from it.

If the value adopted for N is wrong, the period of 19 years would be discovered in the nonpolar change in latitude, but this might have a cause which is shown in Chapter 19. Our problem was to make a check, as complete as possible, of the theory of the Earth's rotation.

This is not only restricted in determining N (the coefficient of the term in obliquity), but we will also examine if there exists a phase of deceleration, and if the observations verify the ratio of the axes of the ellipse of nutation. However, intense research has been carried out on these matters, but no definite answer has been given.

The theory of precession and nutation leads to concrete relations among the constants of precession and nutation and the coefficients from all the other terms in the formula for nutations and right ascension and declination. So, if the constants for precession and nutation are known from observations, all the other coefficients are determined by accurate computation. /180
There was no doubt about the accuracy of the theory, and there were no indications, in trying to determine them, of other coefficients different from the observations. But this kind of determination, as we have already said, is of interest if it constitutes a way to check the theory. Recently Sekiguchi, Morgan and Popov independently tried to determine the range of the 14-day term, from a variety of latitude observations. The basis for such a determination is the following. If the range from a 14-day term obtained from the theory differs from the correct one, a term $\alpha \sin(2☾' - \alpha)$ appears in the change of latitude, where ☾' is the mean length of the moon and α is the mean right ascension of the pair of stars (or the right ascension of a zenith star). It must be noted, however, that the diurnal term of the moon in Oppolver's expression for forced latitude change has exactly this formula. This leads to an uncertainty in the representation of the results of analysis of the observations. A possible assumption, made by Morgan, is that the theoretical value of the coefficient of this term is not subject to correction, and its difference from the observed value can be considered as a correction of the 14-day term. But certainly, we can not do that. Oppolver took the expression for forced change of latitude as one of the results of the theory of the rotation of a rigid Earth. In Chapter 1, we prove the /181
corresponding problem, starting from the hypothesis that the

Earth is an elastic body, but since our intention is to check this hypothesis, we can use the theoretical expression for forced change of latitude only for comparison with the observations, and not for correcting the results of these observations. The uncertainty mentioned above can be overcome in the following way. The formulas for precession and nutation describe the motion in space from the angular momentum of the Earth. In Chapter 1, we will see that these formulas do not change by changing the hypothesis about the mechanical properties of the Earth. This means that if a term $\alpha \sin(\text{☾} - \alpha)$ is discovered in analyzing the observations for latitude, this is recognized as something corresponding to the diurnal term of the moon. Thus, the observations must answer the following questions:

Does forced polar motion take place according to the theory of an elastic Earth?

This matter is analyzed in the third chapter. We will not attempt to construct a new theory on the basis of other hypotheses about the mechanical properties of the Earth. However, we will consider briefly the hypothesis of the liquid core of the Earth in the last chapter, where we use the results of the previous chapters in order to derive certain conclusions about the interaction between the core and the crust.

The initial conditions for all the above computations will be the result of observations of the pairs of stars by Tolcott's method. We call the center of the arc of a great circle connecting two stars the center of the pair. When we say "declination of a pair," we mean the declination of the above center (exactly /182 as in the expression "declination of the sun," etc.). Obviously, it is equal to 1/2 the sum of the declinations of the stars which constitute the pair.

We similarly define the terms "zenith distance" and "right ascension" of a pair. Sometimes we will use abbreviations, when this does not lead to any confusion.

1. About classical theory.
2. How it was abandoned (Chandler).
3. Comparison of elastic-rigid.
4. About the value of N.
5. Jeffrey → core.
6. 14-day term → elastic or rigid?

Summarizing what we have said in the introduction to the newer theory, we can observe the following. The classical theory based on the assumption of a rigid Earth is not valid any more, and this was proved from the period Chandler found. The newer theory accepts that the Earth is an elastic body. This theory seems to be valid. Indeed, Newcomb, taking into account the elastic deformation of the Earth, showed that because of this we have the increase in period found by Chandler.

After that, Schweydar showed that this increase in period is the only manifestation of the assumption about the elasticity of the Earth. Therefore, in developing the theory, such an influence must be shown, and also we must try to find other possible indications of this hypothesis. Afterward, we will compare our results with those of the observations in order to draw conclusions for the validity of the theory. We must /183 therefore examine whether forced polar motion takes place according to the assumption of an elastic Earth.

We have already noted that we have certain disagreements, as, for example, the value of N (constant of nutation). Therefore, it will be necessary to improve our assumptions. For that, we will consider the relationship between the crust and core of the Earth.

When we say that the Earth deforms elastically, we essentially imply small deformations obeying Hook's law. But this deformation causes change in the inertia. And exactly this change in the inertia has as a result an increase in the period. So our main concern is to find the change of inertia tensor due to elastic deformation and to substitute this new tensor into the equations of motion. Exactly this is carried out below.

9.1 [sic] Deformations

Before we present the newer theory, it is necessary to mention a few things about the concepts which made the newer theory necessary. That is, about the general deformation of the Earth due to the gravitation fields of the Moon and the Sun (of a tidal nature). The following are also a first introduction to the theory of plasticity.

9.1.1. General Deformation of the Earth

The real difficulty comes from the fact that our research object is the Earth's deformation. For cases where the wobble is on the order of 1 year or less, and for a purely elastic deformation, the description of the deformation is made by /184 introducing the Love numbers. With an appropriate choice, a variety of problems can be solved with notable ease. But this is a misleading situation, because the corresponding elasticity problems have been solved in such a way as to make the Love numbers appropriate.

Considering the general problem, we could tolerate certain deviations from the theory of elasticity. In any case, the choice of a purely inelastic model for the Earth includes many assumptions, and the exact solutions are not easily attainable and not worth the effort of obtaining them at the present state of the art. Kelvin assumed that the Earth behaves elastically for deformations of a small period, while Darwin assumed that the Earth behaves plastically only for small forces. Today, our standpoint, with regard to the stars, remains the same. Only for some deformation problems, small progress has taken place. In the previous chapter, we presented a few points about the theory of elasticity. As far as plastic deformations are concerned, the relations are relatively simple, if the elastic stress is small. Thus, we can have deformations remaining, but the elastic strain is still constant and small. This is the easiest case in the theory of plasticity.

However, in the case of the Earth, the total elastic strain is often very large due to hydrostatic pressure. For this reason, we assume that the strain is composed of two parts, i.e. of one large initial strain due to the self-attraction of the material elements, and one small strain due to small changes in the forces acting on the system. Of course, we are interested only in the superimposed small strain. The value of the plastic deformation is /185

$$d_{ij} = d_{ij} - \frac{d\varepsilon_{ij}}{dt} \tag{1}$$

where d_{ij} is the total value of the deformation and ε_{ij} is the elastic strain. A material is plastically deformed if $\alpha_{ij} \neq 0$. The total applied stress in the Earth's interior is

$$p_{ij} = -p\delta_{ij} + \tau_{ij} + s_{ij} \tag{2}$$

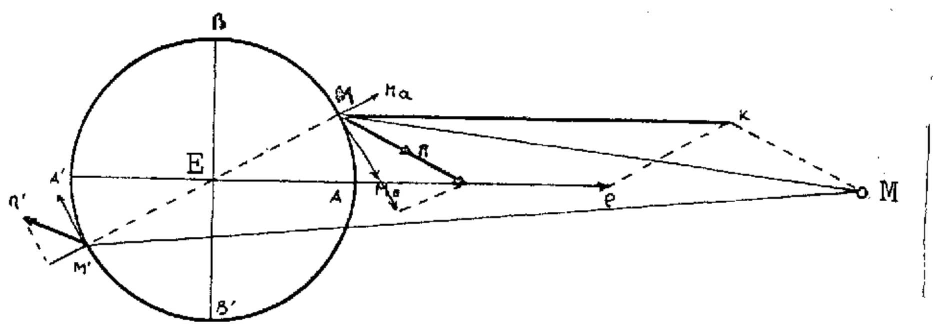
where p is the hydrostatic pressure.

$\tau_{i\lambda}$ is the elastic stress = $\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$, λ, μ = elastic constants. $s_{i\lambda}$ is the stress due to friction. This depends on the model of the Earth. The only force on the Earth which is in hydrostatic equilibrium is the hydrostatic pressure. The distribution of the density at any point in the Earth defines the size of this pressure. Today, we initially consider the Earth in hydrostatic equilibrium, and then we examine possible deviations by relation (2).

Finally, in order to obtain the force-deformation relations, it is necessary to know the plastic deformation. The exact nature of this dependence is not yet fully known, and it depends on experimental research. Therefore, the problem of plastic deformation has not yet been completely solved.

9.1.2. Tidal Deformations

The surface of the lakes and seas of the Earth is subject to periodic rise and fall in an interval slightly greater than 1 day. Newton interpreted this phenomenon as a result of the gravitational force applied to the Earth by the Moon, mainly in combination with the gravitational force of the Sun. The solid /186 crust of the Earth and the atmosphere of the Earth are subject to an analogous influence, on a smaller scale. Tides are due to the difference of the gravity of the Moon on the Earth's surface from the gravity of the Earth's center. The result of this difference in gravity is the rise in the directions connecting the Earth with the Moon, and fall in the directions perpendicular to the former.



Consider the positions of the Earth and the Moon (E,M). If MM is the gravitational force of the Moon per unit mass at point M and EP is the corresponding unit at point E, and if we analyze MM in two components, one equal to EP and the other to MΠ, then MΠ causes the water to rise and brings it toward A. We analyze MΠ in the radial component Ma and the tangent ME (circumferential). Then Ma reduces the intensity of gravity at A, A' and increases the intensity of gravity at B, B' and ME moves the water toward A, A'. Therefore, we have rise at A, A' and fall at B, B', and the water mass tends to take the shape of an ellipsoid by revolution around the direction connecting the Earth with the Moon. The tidal force (the difference of the gravity on the surface and in the center of the Earth) is proportional to the mass and inversely proportional to the cube of the distance. This can be proved as follows. If M is the mass of the Moon with respect to the mass of the Earth, r its distance measured in radii of the Earth, then the acceleration at E will be $g\frac{M}{r^2}$ and at A, $g\frac{M}{(r-1)^2}$. Therefore, the tidal force F will be /187

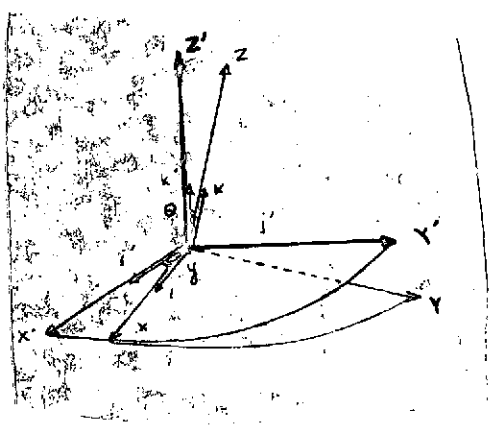
$$F = gM\left[\frac{1}{r^2} - \frac{1}{(r-1)^2}\right] = gM\frac{r^2-2r+1-r^2}{r^2(r-1)^2} = -gM\frac{2r+1}{r^2(r-1)^2} \simeq \frac{-2gM}{r^3}$$

If the Sun lies on the same straight line connecting the Earth with the Moon, its force of gravity is added to the one of the Moon. The gravity of the Moon and Sun tend to decelerate gradually the rotation of the Earth.

9.2. Theory of the Rotational Motion of an Elastic Body /188

9.2.1. Derivation of the Equations of Motion from the Angular Momentum of the Earth

Consider a right-handed fixed coordinate system X', Y', Z'. The plane X'OY' coincides with the plane of the ecliptic at an initial time, and the axis OX' passes through the point γ of the vernal equinox of this time.



In this system, we denote the unit vectors as $\overline{i}'$, $\overline{j}'$, $\overline{k}'$. We also define a second coordinate system as follows: The axis OZ is in the direction of a unit vector $\overline{K}$ and the axis OX along the direction of the vector i, where

$$i = \frac{K \wedge K'}{\sin\theta}$$

(Note: θ = obliquity; Ψ = length measured from one position), and θ is the angle between K, K'. We call the angle between OX' and OX y.

The cross product of two vectors is a vector normal to the plane of the other two with norm equal to the product of the norms of the two vectors times the sine of their angle:

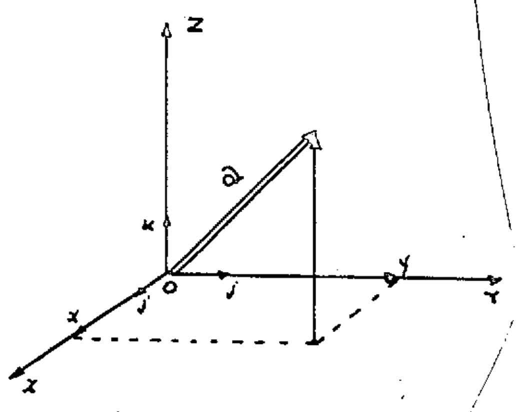
$$K \wedge K' = i \cdot \sin\theta = K \cdot K' \cdot \sin\theta$$

i.e., the vector of norm $i = K \cdot K'$ lies on the plane X'OY' in such a way that the three vectors K, K', $K \wedge K'$ constitute a right-handed coordinate system.

Any vector can be written in the following form:

$$\bar{\lambda} = x'i' + y'j' + z'k' = xi + yj + zk = k a_\varphi + k' a_\psi - i a_\vartheta \qquad (1.1)$$

where x', y', z' and x,y,z are the projections of a vector $\bar{\lambda}$. On the axes of the two systems $\alpha\phi/\phi\omega$ and α_ϕ, α_ψ, $-\alpha_\theta$ are the components of the vector in the three directions K, K', i.



$$\bar{a} = x\bar{i} + y\bar{j} + z\bar{k}$$

From Eq. (1.1) it is easy to find the expression for the projections of the vector on the two systems. For this purpose, we use the following table of direction cosines of the axes of the two systems. /189

The table is constructed by using the spherical triangle defined from the axes of a sphere having as center the point 0 of the system; for example, in order to find the expression for x' in terms of α_ϕ, α_ψ, $-\alpha_\theta$, we take the scalar product of (1.1) with

$$i' \rightarrow x' = i'\bar{\lambda} \rightarrow$$

$$x' = \bar{i}'\bar{\lambda} = i'k a_\varphi + i'k' a_\psi - i i' a_\theta = -\sin\vartheta \sin\psi \, a_\varphi - \cos\psi \, a_\theta$$

	$\bar{i}'$	$\bar{j}'$	$\bar{k}'$
$\bar{i}$	$\cos\psi$	$\sin\psi$	0
$\bar{j}$	$-\cos\vartheta \sin\psi$	$\cos\vartheta \cos\psi$	$-\sin\vartheta$
$\bar{k}$	$-\sin\vartheta \sin\psi$	$\sin\vartheta \cos\psi$	$\cos\vartheta$

We call $\bar{G}$ the angular momentum (expressed in terms of the angular velocity). Then $\bar{G} = K|G|$. Therefore, the projections of $\bar{G}$ on the fixed axes are given from the relations

$$x' = -G\sin\vartheta\sin\psi, \quad y' = G\sin\vartheta\cos\psi, \quad z' = G\cos\vartheta \tag{1.2}$$

We call the angular velocity of the rotation $\bar{\Omega}$. Then, from the rate of change of the angular momentum, we obtain

$$K|G'| + \bar{\Omega} \wedge \bar{G} = \bar{L} \tag{1.3}$$

where $\bar{L}$ is the torque of the externally applied forces.

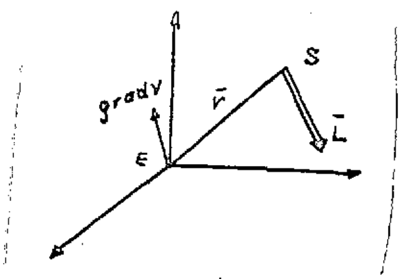
Assume that $\bar{r} = x'\bar{i}' + y'\bar{j}' + z'\bar{k}'$ is the position vector of the celestial body (Sun or Moon). The force applied on this body by the Earth is V (grad v), where V depends on the direction /190
of the principal axes of inertia.

$$\bar{L} = -\bar{r} \wedge \mathrm{grad}\, V \tag{1.4}$$

where

$$V = -\frac{3}{2}\frac{fM}{r^5}\bar{r}\cdot I\bar{r}$$

where M is the mass of the celestial body, f the gravity constant, and I the inertia tensor of the Earth. Consider the following figure.



As usually the force is related to the potential (of energy, or of gravity, etc.) by a relation of the form

$$\bar{F} = -\frac{dV}{dr} \quad \text{but} \quad \frac{dV}{dr} = \mathrm{grad}\, V\cdot\bar{r}_0 = \tag{1}$$

Therefore, the vector of the external torque will be

$$\bar{L} = -\bar{r} \wedge \mathrm{grad}\, V$$

From $V = -\frac{3}{2}\frac{fM}{r^5}\bar{r}\cdot I\bar{r}$ we will have:

$$\operatorname{grad} V = -\frac{3}{2} f M \left[\frac{1}{r^5} \operatorname{grad} (\bar{r} \cdot I\bar{r}) + \bar{r} \cdot I\bar{r} \ \operatorname{grad} \frac{1}{r^5} \right] \tag{1.5}$$

Since grad $\frac{1}{r^5}$ is in the direction of $\bar{r}$, we can omit the second term (the inner product does not contribute). It is also true that grad $(\bar{r} \cdot I\bar{r}) = 2Ir$. So if we substitute in Eq. (1.4), we have:

$$\bar{k} \dot{G} + \bar{\Omega} \wedge \bar{G} = \frac{3fM}{r^5} \bar{r} \wedge I\bar{r} \tag{1.6}$$

Since the system $x\psi z$ rotates, the angles θ, ψ change. Therefore, we can write the angular velocity of this frame as the vector sum of the two angular velocities, i.e.

$$\bar{\Omega} = \bar{k}' \dot{\psi} - \bar{i}' \dot{\vartheta} \quad \text{because} \quad \begin{matrix} \omega_1 = \dot{\psi} \rightsquigarrow \bar{\omega}_1 = \dot{\psi} \bar{k}' \\ \omega_2 = \dot{\vartheta} \rightsquigarrow \bar{\omega}_2 = -\bar{i} \dot{\vartheta} \end{matrix}$$

(see [reference left out of original]).

$$\bar{k} \dot{G} + \bar{\Omega} \wedge \bar{G} = \frac{3fM}{r^5} \bar{r} \wedge I\bar{r}$$

we multiply both members (1) by i

$$\rightsquigarrow \bar{i}\bar{k} \dot{G} + \bar{i}\bar{\Omega} \wedge \bar{G} = \bar{i} \frac{3fM}{r^5} \bar{r} \wedge I\bar{r} \qquad \bar{i} \perp \bar{k} \rightsquigarrow \bar{i}\bar{k} = ik \cos 90° = 0$$

$$\rightsquigarrow \bar{i}\bar{\Omega} \wedge \bar{k} \bar{G} = \frac{3fM}{r^5} Ir \ (\bar{i} \wedge \bar{r})$$

but $\bar{i} \wedge \bar{k} = \bar{r}' \omega\vartheta \rightsquigarrow \bar{\Omega} (\bar{i} \wedge \bar{k}) = \bar{\Omega} \bar{k}' \omega\vartheta = \omega_k \quad \omega\vartheta = \dot{\psi}$

$$\Rightarrow G \omega\vartheta \dot{\psi} = \frac{3fM}{r^5} Ir \ (i \wedge \bar{r})$$

Similarly, the other.

Therefore, /191

because

$$\bar{\Omega} \wedge \bar{G} = G (\bar{k}' \wedge k\dot{\psi} - \bar{i} \wedge k\dot{\vartheta}) = G (-\bar{i} \sin \vartheta \dot{\psi} + \bar{j} \dot{\theta})$$

$$\bar{G} = G\bar{k} \rightsquigarrow \bar{\Omega} \wedge \bar{G} = G\bar{k} \wedge (\bar{k}'\dot{\psi} - \bar{i}\dot{\vartheta}) = G (\bar{k}' \wedge \bar{k}\dot{\psi} - \bar{i} \wedge \bar{k}\dot{\vartheta}$$

Using this result, we take the inner scalar product of both sides of Eq. (1.6) with the vectors $-\bar{z}$, k and $\bar{k}'$ in rotation. Then we obtain (1):

$$
\left.\begin{aligned}
G \sin \vartheta\, \dot{\psi} &= \frac{sfM}{r^5}\, i\, (\bar{r} \wedge Ir) = -\frac{sfM}{r^5}\, Ir\, (\bar{\imath} \wedge \bar{r}) \text{ times } \bar{\imath} \\
\dot{G} &= \frac{sfM}{r^5}\, \bar{k}\, (\bar{r} \wedge Ir) = \frac{sfM}{r^5}\, Ir\, (\bar{k} \wedge \bar{r}) \text{ times } \bar{k} \\
\dot{G} \cos \vartheta - G \sin \vartheta\, \dot{\theta} &= \frac{sfM}{r^5}\, \bar{k}'\, (\bar{r} \wedge Ir) = \frac{sfM}{r^5}\, Ir\, (\bar{k}' \wedge \bar{r}) \text{ times } \bar{k}'
\end{aligned}\right\} \qquad (1.7)
$$

The last expression was found by the law of transposition of the triple inner product. Essentially, the above equations give the projections of L on the system $\bar{L}$, $\bar{K}$, $\bar{K}'$, where L was substituted by its causal expression.

9.2.2. Change of the Inertia Tensor Due to the Deformation of the Earth

We will find the expression for the inertia tensor of the Earth, as the shape of the Earth changes under the influence of a force of tidal nature. We will consider the ellipsoidal shape of the undeformed Earth (when no force is acting on it) as a biaxial ellipsoid in order to assume that the moment of inertia of the Earth with respect to any equinoctial axis is A. We take the principal axes of this ellipsoid as the axes of an auxiliary coordinate system, and we denote by α, β, γ the cosines of the angles that the axis OZ*(1) of this system forms with the axes of the principal system. The inertia tensor of the Earth in the auxiliary system (system of principal axes, because the ellipsoid was taken ‡ diagonal form) is a diagonal of the form

$$
\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}
$$

(diagonalized)

And it is easily shown that in the system XOY it is (the inertia tensor) AE + (C − A)P (1.8) where E is the Earth tensor and the P tensor is given by OX. /192

$$
P = \begin{Bmatrix} \alpha\alpha & \alpha\beta & \alpha\gamma \\ \beta\alpha & \beta\beta & \beta\gamma \\ \gamma\alpha & \gamma\beta & \gamma\gamma \end{Bmatrix} \qquad (1.9)
$$

Assume now that the motion of the Earth stops. If the Earth were in reality a fluid body (geoid) (a hypothesis usually made in the theory of the shape of the Earth), this would give a spherical shape, but since the Earth as something concrete, or even its crust, is an elastic body, it would remain a spheroid in the absence of rotation, having less pressure than it really has. In this case, we call the inertia tensor I_O and the difference between the polar and the equinoctial moment of inertia

$$(1 - K)\ (C - A)$$

In fact, the Earth rotates not around the polar axis of the tensor I_O but around the instantaneous axes which change con- their position continuously. Along this axis, the elastic Earth contracts with centrifugal force. Therefore, we write the total inertia tensor

$$I = I_O + I_\omega$$

where the tensor I_ω takes the diagonal form in the auxiliary coordinate system, the axis OZ* of which coincides with the vector of the angular velocity $\overline{\omega}$, i.e. the velocity of the axes OZO. The difference among the diagonal elements of this tensor is

$$K(C - A)$$

If the density and the elastic properties of the interior of the Earth are functions of the distance from the center only, then it is proved from the theory of the Earth's tidal deformation that this difference is proportional to the potential of the centrifugal force $V_\omega = -(1/2)\eta^2 W^2$ where η is the modulus (the coefficient, modulus) of the angular velocity of the rotation of the Earth and W_2 a spherical harmonic of the second order. We can now write the angular momentum $\overline{G}$ as follows: /193

$$\overline{G} = (I_O + I_\omega)\overline{\omega}$$

and using (1.8), we obtain:

$$\bar{G} = \left\{(1-K)\left[AE + (C-A)P\right] + K\left[AE + (C-A)P\omega\right]\right\}\bar{\omega} \qquad (1.10)$$

We denote the direction cosines of the polar axis of tensor I_O in this system by α_O, β_O, γ_O. For the whole period for which we have information for polar motion, the α_O and β_O never exceeded $2 \cdot 10^{-6}$ and the γ_O was no more than 10^{-11} different from 1. Therefore, we shall neglect the squares and products of α_O and β_O and we will put γ_O equal to 1. Then the tensor P_O (corresponding to P) takes the following form:

$$P_0 = \begin{pmatrix} 0 & 0 & \alpha_0 \\ 0 & 0 & \beta_0 \\ \alpha_0 & \beta_0 & 1 \end{pmatrix}$$

We denote the cosines of the angles between $\bar{\omega}$ and the axes of the system XYZ by α', β', γ'. They have the same formula as the one of α_0, β_0, γ_0, and we can write the matrix P_ω simply by changing the indices. From (1.10) we obtain by projection on the x-axis

$$(1-\kappa)\left[A\omega_x + (C-A)\alpha_0\,\omega_z\right] + \kappa\left[A\omega_x + (C-A)\alpha'\omega_z\right] = 0$$

Denoting (C-A)/A by α^2 and $\omega_x/\omega_2 = \alpha'$, we find by rearranging that /194

$$\alpha' = -\frac{(1-\kappa)\,\alpha^2}{1+\kappa\alpha^2}\,\alpha_0 \qquad (1.11)$$

The projection on the axis of y gives a corresponding relation between β' and β_0. In the denominators, we can neglect the second term. So we can write

$$\alpha' = -(1-\kappa)\alpha^2\alpha_0 \quad , \quad \beta' = -(1-\kappa)\alpha^2\beta_0 \qquad (1.12)$$

Therefore, the tensor P_ω can be written in the form

$$P_\omega = \begin{pmatrix} 0 & 0 & -(1-\kappa)\alpha^2\alpha_0 \\ 0 & 0 & -(1-\kappa)\beta^2\beta_0 \\ -(1-\kappa)\alpha^2\alpha_0 & -(1-\kappa)a^2\beta_0 & 1 \end{pmatrix} \qquad (1.13)$$

We shall now consider the deformation of the Earth due to the action of the tidal force of the Moon and the Sun. Calculating the change in the elements of tensor I, which occurs because of this deformation, we take as spherical the initial shape of the Earth, and so we can assume that the inertia tensor of the undeformed Earth is AE, where E is the identity tensor. Then, for any coordinate system, the axis OZ* of which coincides with the line OO_1 passing from the center of the Earth P and the changing system, the inertia tensor of the Earth takes a diagonal form, and the difference between the principal moments of inertia is:

$$K_0 \quad (C-A) \frac{V}{V_\omega}$$

where V is the potential of the tidal force. For the Sun, we use the symbol V_1, and for the Moon, V_2. The coefficient K is substituted by the coefficient K_0 in the following. The tide of the ocean is due to the change in centrifugal force, and it can be considered as static. Hence, the change in the inertia tensor is related to both phenomena, i.e. the tide of the ocean and the deformation of the solid sphere. /195

The diurnal and half-diurnal tides of the ocean caused by the action of gravity can not be considered as static, and therefore their amplitudes can not be related with simple proportions to the potential of the changing force. In the above (following) work, we will have to do with diurnal tides initially, because only these change the inertia products of the Earth. It is known that the diurnal tide practically does not exist in the oceans, and for this reason in the calculation of change in moments of inertia in the case considered, it is necessary to take account only of the tide of the body (solid crust) and to use the adjusted coefficients which we obtained after we disregarded the oceanic tide. This coefficient can be denoted by K_0.

From the expressions

$$V_1 = \frac{3}{2} \, f \frac{M_1}{r_1^3} \, W_2 \quad , \quad V_2 = \frac{3}{2} \, f \frac{M_2}{r_2^3} \, W_2$$

we obtain the following values for the difference between the principal moments of inertia of the deformed Earth:

$$C_1 - a_1 = -\frac{3fM_1}{4^2 r_1^2} \, K_0 \, (C-A) \, , \quad C_2 - a_2 = -\frac{3fM_2}{4 r_2^3} \, K_0 \, (C-A) \qquad (1.14)$$

Passing now to the case of a rotating Earth subject to the tidal influence of the Sun and Moon, we can make two additions to the expression for the inertia tensor, so that it takes the form

$$I = I_0 + I\omega + I_1 + I_2 \qquad (1.15)$$

where

$$I_1 = (c_1 - a_1)\,P_1 \quad , \quad I_2 = (c_2 - a_2)\,P_2 \qquad (1.16)$$

$$P_1 = \frac{1}{r_1^2}\begin{pmatrix} x_1 x_1 & x_1 y_1 & x_1 z_1 \\ y_1 x_1 & y_1 y_1 & y_1 z_1 \\ z_1 x_1 & z_1 y_1 & z_1 z_1 \end{pmatrix} \quad , \quad P_2 = \frac{1}{r_2^2}\begin{pmatrix} x_2 x_2 & x_2 y_2 & x_2 z_2 \\ y_2 x_2 & y_2 y_2 & y_2 z_2 \\ z_2 x_2 & z_2 y_2 & z_2 z_2 \end{pmatrix} \qquad (1.17)$$

where x_1, y_1, z_1 and x_2, y_2, z_2 are the coordinates of the Sun and the Moon, and r_1, r_2 the corresponding distances of their centers from the center of the Earth. Now (1.15) can be written /196

$$I = (1-k)\left[AE + (C-A)P_0\right] + k\left[AE + (C-A)P_\omega\right] - \frac{3fM_1}{\eta^2 r_1^3} k_0 (C-A) P_1 - \frac{3fM_2}{\eta^2 r_2^3} k_0 (C-A) P_2$$

or

$$I = AE + (C-A)\left[(1-k)P_0 + kP_\omega - \frac{3fk_0}{\eta^2}\left(\frac{M_1}{r_1^3} P_1 + \frac{M_2}{r_2^3} P_2\right)\right] \qquad (1.18)$$

9.2.3. Equations of Precession and Nutation

The expression for I obtained in the previous chapter must be substituted in (1.6)

$$\bar{K}\dot{G} + \bar{\Omega} \wedge \bar{G} = \frac{3fM}{r^5}\, \bar{r} \wedge I\bar{r}$$

in which we write the right-hand side as the sum of two terms,

$$\frac{3fM_1}{r_1^5}\, \bar{r}_1 \wedge I\bar{r}_1 + \frac{3fM_2}{r_2^5}\, \bar{r}_2 \wedge I\bar{r}_2$$

After substitution, we obtain the following groups of terms where a common factor has been omitted:

$$\frac{M_1^2}{r_1^8}\bar{r}_1 \wedge P_1 \bar{r}_1 + \frac{M_2^2}{r_2^8}\bar{r}_2 \wedge P_2 \bar{r}_2 + \frac{M_1 M_2}{r_1^5 r_2^5}\left(r_1^2 \bar{r}_1 \wedge P_2 \bar{r}_1 + r_2^2 \bar{r}_2 \wedge P_1 \bar{r}_2\right)$$

It is easily shown that

$$r_1^2 \bar{r}_1 \wedge P_2 \bar{r}_1 = (\bar{r}_1 \wedge \bar{r}_2)\ (\bar{r}_1 \cdot \bar{r}_2)$$

If $\bar{v}_1 = \bar{v}_2$, the cross product $\bar{v}_1 \wedge \bar{v}_2 = 0$. Also, $\bar{v}_1 \wedge \bar{v}_2 = -\bar{v}_2 \wedge \bar{v}_1$, and for this reason the last sum of the above term is zero. We can therefore consider separately the changing influence of the Sun and Moon. To reduce the computations, we again restrict our attention to one term of (1.6). Taking account of $\bar{r} \wedge E\bar{r} = 0$, we obtain

$$\frac{3fM}{r^5}\bar{r} \wedge I\bar{r} = \frac{3fM}{r^5}\bar{r} \wedge (I_0 + I\omega + I_1)\bar{r} = \frac{3fM}{r^5}(C-A)\left[(1-k)\bar{r} \wedge P_0 \bar{r} + k\bar{r} \wedge P_\omega \bar{r}\right]$$

Using this expression, we can transform the right-hand side of the equation $AE + (C - A)P$, taking account first of

/197

$$\bar{i} \wedge \bar{r} = \bar{j} z + \bar{k} y, \qquad \bar{k} \wedge \bar{r} = -\bar{i} y + \bar{j} x$$

$$\bar{k}' \wedge \bar{r} = -\bar{i}\,(z \sin\theta + y\cos\theta) + \bar{j}\, x\cos\theta + \bar{k}\, x \sin\theta$$

$$P_0 \bar{r} = \bar{i}\,\alpha_0 z + \bar{j}\,\beta_0 z + \bar{k}\,(\alpha_0 x + \beta_0 y + z)$$

On the other hand

Hence
$$\begin{cases} P_0 \bar{r}\ (\bar{i} \wedge \bar{r}) = yz + \alpha_0 xy + \beta_0 (y^2 - z^2) = yz + \varepsilon \\ P_0 \bar{r}\ (\bar{k} \wedge \bar{r}) = \beta_0 xz - \alpha_0 yz = \varepsilon' \\ P_0 \bar{r}\ (\bar{k}' \wedge \bar{r}) = xz\sin\theta + z\cos\theta\,(\beta_0 x - \alpha_0 y) + \sin\theta\,(\alpha_0 x^2 - \alpha_0 z^2 + \beta_0 xy) \\ \qquad = xz\sin\theta + \varepsilon'' \end{cases}$$

where we denote by ε, ε', ε'' the sums of the terms which include the α_0 and β_0, and therefore they are on the order of 10^{-6}.

Similarly, we obtain

$$P\omega\bar{r}\,(\bar{i}\wedge\bar{r}) = yz - (1-k)\,\alpha^2\varepsilon\,, \qquad P\omega\bar{r}\,(\bar{k}\wedge\bar{r}) = -(1-k)\alpha^2\varepsilon'\,,$$

$$P\omega\bar{r}\,(\bar{k}'\wedge\bar{r}) = xz\,\sin\vartheta - (1-k)\,\alpha^2\varepsilon''$$

Here, the terms including ε, ε', ε" can be neglected. Then the equations (1.7) can be written

$$G\sin\vartheta\dot{\psi} = -\frac{3fM}{r^5}\,(C-A)\,[yz + (1-k)\,\varepsilon]$$

$$\dot{G} = \frac{3fM}{r^5}\,(C-A)\,(1-k)\,\varepsilon'$$

$$\dot{G}\cos\vartheta - G\sin\vartheta\,\dot{\theta} = \frac{3fM}{r^5}\,(C-A)\,[xz\sin\vartheta + (1-k)\,\varepsilon'']$$

In order to make an approximate calculation of the change in angular velocity, we substitute C.n for $\bar{G}$. Then:

$$n\dot{\psi} = -\frac{3fM}{r^5}\left(\frac{C-A}{C}\,\frac{1}{\sin\vartheta}\quad yz + (1-k)\,\varepsilon\right]$$

$$\dot{n} = \frac{3fM}{r^5}\,\frac{C-A}{C}\,(1-k)\,\varepsilon' = \eta\,(1-k)\,(\alpha_0\sin\vartheta\,\dot{\psi} - \beta_0\dot{\vartheta}) \qquad (1.20)$$

$$n\dot{\vartheta} = -\frac{3fM}{r^5}\,\frac{C-A}{C}\left[xz + \frac{1-k}{\sin\theta}\,(\varepsilon'' - \varepsilon'\cos\theta)\right]$$

We remind the reader at this point that α_0 and β_0 are essentially the direction cosines of the polar axis of the Earth, with respect to the XYZ system. These axes have an angular velocity of the Earth around the instantaneous axis of rotation which almost coincides with the direction of $\bar{G}$. /198

Hence, we can approximately obtain, by appropriately choosing the epoch from which time is computed

$$\alpha_0 = \sigma\cos nt \qquad \beta_0 = \sigma\sin nt$$

$\sigma = 10^6$ (order of magnitude), and then it will be

$$\Delta n = (1-k)\,n\sigma\int(\dot{\psi}\sin\vartheta\,\cos nt - \dot{\theta}\sin nt)\,dt$$

Since $\dot{\theta}$ and $\dot{\psi}$ change slowly compared to nt, we will consider them as constant in the solution of the above integral equation (by integration). We will have:

$$\Delta n = (1-k)^n \epsilon \, (\dot{\psi} \sin\theta \, \sin nt + \dot{\theta} \cos nt)$$

From the terms in the expression for $\dot{\psi}$ and $\dot{\theta}$, the greater seems to be the constant term $\dot{\psi}$, which is the Moon-Sun precession. In computing this term, we only consider a computation of the amplitude of the oscillation of Δn. Using the values

$$1-k = 0.7, \quad \epsilon = 10^{-6}, \quad \sin\theta = 0.4$$

$\dot{\psi}$ = 50" per year, or $6.8 \cdot 10^{-7}$ rad per day, we obtain $\Delta n = 1.9 \cdot 10^{-13}$ rad per day. From this computation, we can conclude the following:

1. The angular velocity of the diurnal rotation of the Earth and the magnitude of $\overline{G}$ can be taken as constants (while in the theory of the rigid body, they were constants in any case).

2. ε, ε', ε'' are small quantities, all of the same order, and for this reason we can, with satisfactory accuracy, substitute Eqs. (1.20) by the following:

$$\begin{aligned} \dot{\psi} &= -\frac{3fM}{nr^5} \frac{C-A}{C} \frac{1}{\sin\theta} yz \\ \dot{\theta} &= \frac{3fM}{nr^5} \frac{C-A}{C} xz \end{aligned} \qquad (1.21)$$

ψ, θ = angles between the two figures.

These are the usual equations of precession and nutation. /199
The method of integration is well-known.

We observe that the elastic deformation of the Earth does not influence the motion in space of the vector $\overline{G}$, the angular momentum of the Earth. Moreover, the equations of motion of $\overline{G}$ are practically unchanged for any hypothesis concerning the interior of the Earth, because for all acceptable hypotheses, the tidal deformation of the Earth has so small an influence on the form of the inertia ellipsoid, that its result can always be incorporated in the torques of the external forces.

9.2.4. Differential Equations of Motion of the Vector of Angular Momentum with Respect to the Earth

In order to complete the solution to the problem of the motion of the elastically deforming Earth, we must find the motion of the principal axes of the tensor I_o with respect to the system XYZ, or vice versa, the motion of the system XYZ with respect to the principal axes of the tensor I_o which we denote by OY_o, OX_o, OZ_o.

These axes rotate with an angular velocity $\bar{\omega}$. Hence, we can write

$$\dot{G} + \bar{\omega} \wedge \bar{G} = \bar{L} \quad \text{but} \quad \bar{L} = \bar{o} \wedge \bar{G} \Rightarrow$$

$$\Rightarrow \quad \dot{G} + (\bar{\omega} - \bar{o}) \wedge G = 0 \tag{1.22}$$

The angular velocity $\bar{\omega} = I^{-1}\bar{G}$ where I^{-1} is the inverse of the tensor I. In order to compute the entries of this matrix, we can use the formula $(I^{-1}_{k\ell}) = Q_{\ell k}/D$ where the indices k and ℓ denote the columns and rows of the matrix, $Q_{\ell k}$ is the minor matrix of the entry $(I_{\ell k})$ multiplied by $(-1)^{k+\ell}$ and D is the determinant of the matrix.

In this formula, and to the first order of the small quantities

/200

$$\sim I^{-1} = \frac{E}{A} - (1-\kappa)\frac{C-A}{CA}P_o + \frac{1}{AC}(I_\omega + I) =$$

$$= \frac{E}{A} - (1-\kappa)\frac{C-A}{CA}P_o \frac{\kappa(C-A)}{CA}P_\omega + \frac{3fM}{n^2\nu^3}K_o\frac{C-A}{CA}P_1 \tag{1.23}$$

We now project the vectors of Eq. (1.22) on the axes of the system X_o, Y_o, Z_o and we write the tensors in the form

$$P_o = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P_\omega = \frac{1}{n}\begin{pmatrix} 0 & 0 & \omega_x \\ 0 & 0 & \omega_Y \\ \omega_x & \omega_Y & \omega_z \end{pmatrix}$$

The tensor I takes the same form as in the system XYZ, but it is understood that in order to compute its entries we use the coordinates of the celestial body in the system $X_0Y_0Z_0$. We write Eq. (1.22) as follows:

$$\dot{\bar{G}} + I^{-1}\bar{G} \wedge \bar{G} - \underline{\Omega} \wedge \bar{G} = 0 \tag{1.24}$$

We consider separately the sum of the terms

$$\frac{3fM}{n^2 r^3} k_0 \frac{C-A}{AC} P_1 \bar{G} \wedge \bar{G} - \bar{\Omega} \wedge \bar{G} = \bar{F}$$

and we note that

$$\bar{\Omega} \wedge \bar{G} = \bar{L} = \frac{3fM}{r^5} \bar{r} \wedge I\bar{r}$$

where approximately

$$\bar{\Omega} \wedge \bar{G} = \frac{3fM}{r^5} (C-A) (\bar{i}_0 z_0 y_0 - \bar{j}_0 x_0 z_0)$$

On the other hand, we have the following approximate expression:

$$P_1 \bar{G} \wedge \bar{G} = \frac{C^2 n^2}{r^2} (\bar{i}_0 z_0 y_0 - \bar{j}_0 x_0 z_0)$$

Hence

$$\bar{F} = \frac{3fM}{r^5} (C-A) \left(\frac{C}{A} k_0 - 1\right) (\bar{i}_0 z_0 y_0 - \bar{j}_0 x_0 z_0) = -\left(1 - \frac{C}{A} k_0\right) \bar{\Omega} \wedge \bar{G}$$

and because C/A is approximately equal to 1, we have

$$\bar{E} = -(1 - k_0) \bar{\Omega} \wedge \bar{G} \tag{1.25}$$

The vector $P_\omega \bar{G}$ is satisfactorily close to the direction of $\bar{G}$, and we can take to the first approximation

$$P_\omega \bar{G} \wedge \bar{G} = 0$$

Then, in the expression for the tensor I^{-1}, we can neglect the third term and certainly the first, because $E\bar{G} = \bar{G}$. Taking all these into account, we can write Eq. (1.22) as follows:

/201

$$\dot{\bar{G}} - (1-K)\frac{C-A}{CA} p_0 \bar{G} \wedge \bar{G} - (1-K_0)\bar{\Omega} \wedge \bar{G} = 0 \qquad (1.26)$$

We write the equations obtained by projection on the axes of the system $X_0Y_0Z_0$ as:

$$\begin{aligned} &\dot{X} + (1-K)\alpha^2 n Y - (1-K_0) Cn\Omega_y = 0 \\ &\dot{Y} - (1-K)\alpha^2 n X - (1-K_0) Cn\Omega_x = 0 \\ &\dot{Z} = 0 \end{aligned} \Longrightarrow \bar{G} \parallel Z_0 \text{ because } G \ll \dot{G} \qquad (1.27)$$

where by x, y, z we denote the projections of the vector $\bar{G}$ on the axes of the system. It remains to express the Ω_X, Ω_Y in terms of $\dot{\psi}$ and $\dot{\theta}$ which are the projections of the angular velocity on the axes OZ' and OX. The position of the system of axes $X_0Y_0Z_0$ relative to the XYZ is defined by the Euler angles u, v, θ, where u = $X\hat{O}M$ is measured on the plane XOY, v = $M\hat{O}X_0$ on the plane X_0OY_0 (OM = intersection of the planes Y_0OX_0 and XOY). Since the angle θ_0 is small, we can construct on the plane XOY the angle $\phi = X\hat{O}N$ = u + v. We take

Then, it will be : $X\hat{O}X_0 = Y\hat{O}Y_0 = \varphi$, $X_0\hat{O}Y = 90-\varphi$ $X\hat{O}Y_0 = 90+\varphi$

$$\longrightarrow \Omega_X = -\sin\theta \sin\varphi\,\dot{\psi} - \cos\varphi\,\dot{\theta}, \quad \Omega_Y = -\sin\theta\cos\varphi\,\dot{\psi} + \sin\varphi\,\dot{\theta}$$

Substituting these values in (1.27), we finally find

$$\begin{aligned} &\dot{X} + (1-K)\alpha^2 n Y = Cn(1-K_0)(-\sin\theta\cos\varphi\,\dot{\psi} + \sin\varphi\,\dot{\theta}) \\ &\dot{Y} - (1-K)\alpha^2 n X = Cn(1-K_0)(\sin\theta\sin\varphi\,\dot{\psi} + \cos\varphi\,\dot{\theta}) \end{aligned} \qquad (1.28)$$

9.2.5. Integration of the Equations of Relative Motion of the Vector of Angular Momentum

Integrating the equations of relative motion of the vector of angular momentum, we can use these expressions for ψ and θ which are given by the theory of precession and nutation, which is based on the hypothesis that the Earth is a rigid body.

We write the total angular velocity as the sum of the relative angular velocities $\bar{\omega} = \bar{\omega}' + \Omega$. /202

Projecting on the direction of $\bar{G} \rightsquigarrow n = \dot{\varphi} + \dot{\psi}\cos\vartheta$

from which is obtained

$$\Rightarrow \varphi = nt - \int \dot{\psi}\cos\vartheta \, dt = n_1 t \tag{1.29}$$

The more important term in the expression for ψ is the Moon-Sun precession. The change in θ is not important. Hence n_1 is practically constant. Using it in (1.28), we have

$$\begin{aligned} \dot{X} + (1-K)\alpha^2 n Y &= Cn(1-K_0)(-\sin\vartheta \cos n_1 t\, \dot{\psi} + \sin n_1 t\, \dot{\vartheta}) \\ \dot{Y} - (1-K)\alpha^2 n X &= Cn(1-K_0)(\sin\vartheta \sin n_1 t\, \dot{\psi} + \cos n_1 t\, \dot{\vartheta}) \end{aligned} \tag{1.30}$$

The expressions for precession and nutation in longitude and obliquity are

$$\rightsquigarrow \psi = pt + \Sigma N_i \sin\mu_i t \quad , \quad \vartheta = \vartheta_0 + \Sigma M_i \cos\mu_i t$$

Hence $\dot{\psi} = p + \Sigma N_i \mu_i \cos\mu_i t \quad , \quad \dot{\vartheta} = \Sigma M_i \mu_i \sin\mu_i t$ (1.31)

We write $-M_i\mu_i = B_i + B_i' \quad , \quad N_i \mu_i \sin\vartheta = B_i - B_i'$ (1.32)

Then relations (1.30) can be written as follows:

$$\begin{aligned} \dot{X} + (1-K)\alpha^2 n Y &= Cn(1-K_0)\left[-p\sin\vartheta\cos n_1 t - \Sigma B_i \cos(n_1+\mu_i)t + \Sigma B_i' \cos(n_1-\mu_i)t\right] \\ \dot{Y} + (1-K)\alpha^2 n X &= Cn(1-K_0)\left[p\sin\vartheta\sin n_2 t + \Sigma B_i \sin(n_1+\mu_i)t - \Sigma B_i' \sin(n_2-\mu_i)t\right] \end{aligned} \tag{1.33}$$

The solutions of these equations without the right-hand side are:

$$\left.\begin{aligned} X &= u\cos(1-\kappa)\alpha^2 nt + v\sin(1-\kappa)\alpha^2 nt \\ Y &= u\sin(1-\kappa)\alpha^2 nt + v\cos(1-\kappa)\alpha^2 nt \end{aligned}\right| \qquad (1.34)$$

where u, v are constants. Then, in order to obtain the solutions considering also the right-hand side, we first take u, v as functions of time. Then it will be:

$$\left.\begin{aligned} \dot{X} &= \dot{u}\cos(1-\kappa)\alpha^2 nt + \dot{v}\sin(1-\kappa)\alpha^2 nt - (1-\kappa)\alpha^2 n\, Y \\ \dot{Y} &= \dot{u}\sin(1-\kappa)\alpha^2 nt - \dot{v}\cos(1-\kappa)\alpha^2 nt + (1-\kappa)\alpha^2 n\, X \end{aligned}\right|$$

and substituting (1.33), we obtain

/203

$$\left.\begin{aligned} \dot{u} = C_n(1-\kappa_0)\Big[&-p\sin\theta\cos[n_1+(1-\kappa)\alpha^2 n]t - \Sigma B_i\cos[n_1+\mu_i+(1-\kappa)\alpha^2 n]t + \\ &+ \Sigma B'_i\cos[n_1-\mu_i+(1-\kappa)\alpha^2 n]t\Big] \\ \dot{v} = C_n(1-\kappa_0)\Big[&-p\sin\theta\sin[n_1+(1-\kappa)\alpha^2 n]t - \Sigma B_i\sin[n_1+\mu_i+(1-\kappa)\alpha^2 n]t + \\ &+ \Sigma B'_i\sin[n_1-\mu_i+(1-\kappa)\alpha^2 n]t\Big] \end{aligned}\right\}$$

Integrating these expressions, we take sinθ as constant. Integrating and rearranging, we obtain:

$$\left.\begin{aligned} X = {} & u_0\cos(1-\kappa)\alpha^2 nt + v_0\sin(1-\kappa)\alpha^2 nt - \frac{C_n(1-\kappa_0)\,p\sin\theta \,.\, \sin n_1 t}{n_1+(1-\kappa)\alpha^2 n} - \\ & - C_n\frac{1-\kappa_0}{2}\Sigma\frac{N_i\mu_i\sin\theta - M_i\mu_i}{n_1+\mu_i+(1-\kappa)\alpha^2 n}\sin(n_1+\mu_i)t - C_n\frac{1-\kappa_0}{2}\Sigma\frac{N_i\mu_i\sin\theta + M_i\mu_i}{n_1-\mu_i(1-\kappa)\alpha^2 n}\sin(n_1-\mu_i)t \\ Y = {} & u_0\sin(1-\kappa)\alpha^2 nt - v_0\cos(1-\kappa)\alpha^2 nt - \frac{C_n(1-\kappa_0)\,p\sin\theta\cos n_1 t}{n_1+(1-\kappa)\alpha^2 n} - \\ & - C_n\frac{1-\kappa_0}{2}\Sigma\frac{N_i\mu_i\sin\theta - M_i\mu_i}{n_1+\mu_i+(1-\kappa)\alpha^2 n}\cos(n_1+\mu_i)t - C_n\frac{1-\kappa_0}{2}\Sigma\frac{N_i\mu_i\sin\theta + M_i\mu_i}{n_1-\mu_i+(1-\kappa)\alpha^2 n}\cos(n_1-\mu_i)t \end{aligned}\right\}$$

where u_0 and v_0 are arbitrary constants. Equations (1.29) and (1.35) completely define the motion of the system XYZ relative to the $X_0Y_0Z_0$ system, and because the motion of XYZ in space (for example, with respect to X'Y'Z') was found earlier, we can consider that the problem of the rotation of an elastically deforming Earth has been solved.

9.2.6. Polar Motion of an Elastically Deforming Earth

Using the formulas employed for the reduction of observations, we do not take account of the observed declinations of the stars, which are the angular distances from the plane of the instantaneous equator which is perpendicular to the instantaneous axis of the Earth's rotation, because in proving these formulas we do not take into account the result of the diurnal nutation of the coordinates observed.

Starting from formulas (1.21) ($\dot{\psi}$ = ... $\dot{\theta}$ = ...), we describe the motion in space of $\overline{G}$, and not of the instantaneous axis of rotation.

It is concluded that the formula of nutation of declination /204 gives the change in the angular distances of the stars from the plane which is normal to $\overline{G}$. We call it "the plane of the dynamic equator." In computing the latitude according to the observations on the meridian, we use the formula $\phi = \delta + Z$, where Z is the observed zenith distance of the star. Hence, we find the angle between the vertical and the dynamic equator. When we determine the position of the pole from the observations of latitude, we define the position of the pole as the point where a straight line coinciding with the vector $\overline{G}$, not the axis of rotation, intersects the surface.

The difference is not essential, because the angle between $\overline{G}$ and $\overline{\omega}$ does not reach 0".002. The instantaneous axis moves around $\overline{G}$ on a conical surface with a period approximating a sidereal day, and hence the observed declination shows a diurnal term which certainly is not taken into account in computing the phenomenal position.

The reduction of the auxiliary system which is related to the vector $\overline{G}$ (system XYZ) instead of the instantaneous axis has certain advantages in developing the theory of the rotation of the Earth.

For the reduction of the astronomical observation, it is of less importance which system we use. In any event, in this case it is more accurate at the beginning to use the system of the dynamic equator. Then we must define the latitude and the instantaneous pole as we have done previously. We denote the coordinates of the pole by x,y. In accordance with the definition, we have

$$x = \frac{X}{C_n}, \quad y = \frac{Y}{C_n} \tag{1.36}$$

We also write /205

$$\frac{U_0}{C_n} = u_0, \quad \frac{V_0}{C_n} = v_0 \tag{1.37}$$

from (1.35).

$$\Longrightarrow x = u_0 \cos(1-k)\alpha^2 nt + v_0 \sin(1-k)\alpha^2 nt - (1-k_0)\frac{P \sin\vartheta}{n_1 + (1-k)\alpha^2 n} \sin n_1 t -$$

$$- (1-k_0)\left[\sum q_i \sin(n_1+\mu_i)t + \sum q_i' \sin(n_1-\mu_i)t\right]$$

$$y = u_0 \sin(1-k)\alpha^2 nt - v_0 \cos(1-k)\alpha^2 nt - (1-k_0)\frac{P \sin\vartheta}{n_1 + (1-k)\alpha^2 n} \cos n_1 t -$$

$$- (1-k_0)\left[\sum q_i \cos(n_1+\mu_i)t + \sum q_i' \cos(n_1-\mu_i)t\right]$$

where

$$q_i = \frac{N_i \sin\vartheta - M_i}{2} \, \frac{\mu_i}{n_1+\mu_i+(1-k)\alpha^2 n}, \quad q_i' = \frac{N_i \sin\vartheta + M_i}{2} \, \frac{\mu_i}{n_1-\mu_i+(1-k)\alpha^2 n}$$

Considering the sidereal day as the unit of time, $n = 2\pi$. The period of free motion can be found from Ox.

$$\frac{2\pi}{T} = (1-k)\alpha^2 n = (1-k)\frac{C-A}{A} 2\pi \quad \text{hence} \quad 1-k = \frac{A}{C-A} \cdot \frac{1}{T}$$

We have A/(C-A) - 304, T = 433, I - K = 0.72.

The motion of the ocean's water reduces the Love number K. According to Molodenskii, the correction is -0.04. But this is proportional to K $\ne$ of K (there is a formula) → I-K_δ = 0.76.

The information for the computation of the coefficients q_i, q_i'

is taken from Woolard. The arcs are in the note of the Astronomical Yearbook. In Table 1, the values of these coefficients are greater than 0".001. q_i' does not reach this value, so only q_i is given. It must be multiplied by $(I-K_o)$ and substituted in (1.38). If we further multiply the first of these equations by $\cos\lambda$ and the second by $\sin\lambda$ and take the sum, we will find the change in latitude at a point of longitude λ. Since $n_1 t + \lambda = \delta'$ is the sidereal time, we obtain the following expression for the forced change in latitude:

$$\begin{aligned}\Delta\varphi = &- 0''{,}0066 \sin S - 0''{,}0051 \sin(S - 2☾) - 0''{,}0022 \sin(S - 2L) - \\ &- 0''{,}0010 \sin(S - 2☾ - ☊) - 0''{,}0010 \sin(S' - 3☾ - \Gamma') + \\ &+ 0''{,}0009 \sin(S - ☊)\end{aligned} \tag{1.41}$$

TABLE 1. /206

Argument	$N_i \sin\vartheta$	M_i	$\frac{\mu_i}{n_1 - \mu_i + (1-k)\alpha n}$	q_i
☊	6",8586	9",2100	−0,00015	−0,0012
2L	0,5066	0,5522	0,00548	0,0029
2☾	0,0811	0,0884	0,07869	0,0067
2☾−☊	0,0136	0,0183	0,07884	0,0013
2☾−Γ'	0,0104	0,0113	0,12246	0,0013

In this way, we found a synthesis of the small changes in latitude, of a period about equal to the sidereal day. Oppolver first showed that these are consequences of the motion of the rotational axis in space, and he also found their expressions for a rigid Earth.

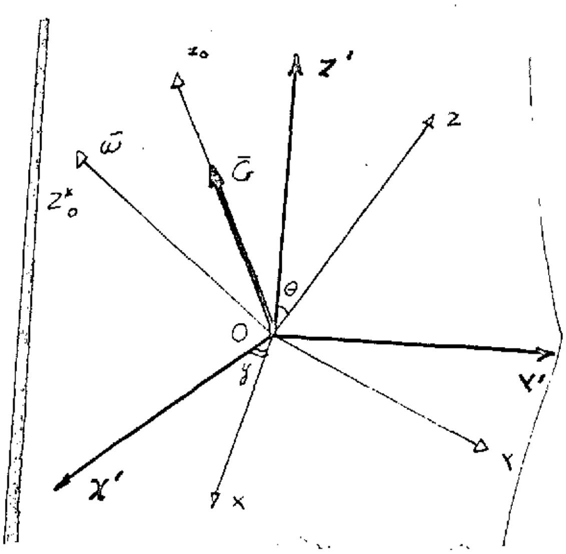
The elastic deformation of the Earth leads to the same relevant reduction of the coefficients of all the terms of Oppolver, because in $\Delta\phi$ the term $I-K_o$ appears everywhere. This and the increase of the period of free nutation are the only evidence of the influence of the deformation of the Earth on its rotational motion, since the deformation has practically no influence on the motion of $\overline{G}$ in space.

9.2.7. A Summary of the Modern Theory

According to what was said in the introduction, we will first find the equations of motion of the system XYZ with respect to a fixed system X'Y'Z'. In the following, we will find the change in the inertia tensor due to the elastic deformation of the Earth. The fine point of this calculation, which underlies the whole elastic consideration of the Earth, is that we can decompose /207 the inertia tensor of the rotating Earth into two tensors I_O, I_ω where I_O corresponds to the undeformed Earth and I_ω to the change due to rotation. This decomposition is valid because the superposition law holds.

Exactly this decomposition constitutes the greatest step of the theory of elasticity, because since we assume elasticity to be valid, under no external cause (force or rotation) will the Earth regain its initial shape. I.e., the tensor I_ω expresses the irrevocable deformation of the Earth due to rotation. If we consider, besides rotation, the influence of the Sun and Moon,

$$I = I_0 + I_\omega + I_1 + I_2$$



In order to substitute the tensor I in the equations of motion, it is necessary to define certain systems.

On XOY → plane of the equator (Z ≡ I)

On X'OY' → plane of the ecliptic (Z' = E')

$\overline{G}$ = vector of angular momentum

Z^*_O = principal axis (direction) $//\overline{\omega}$

Z_O = principal direction of the tensor Z_O

XYZ rotates with $\overline{\omega}$ around Z^*_O and with Ω around X'Y'Z'.

Finally, the instantaneous axis of rotation is close to $\overline{Q}$. Substituting the final form of I in the equations of motion (from the angular momentum of motion), we obtain the equations of recession and

$$\dot{\psi} = -\frac{3fM}{nr^5}\,\frac{C-A}{C}\,\frac{1}{\sin\vartheta}\,yz \quad , \quad \dot{\vartheta} = \frac{3fM}{nr^5}\,\frac{C-A}{C}\,xz$$

We observe that the elastic deformation of the Earth does not affect the motion in space of the vector $\overline{G}$, i.e. $\overline{G}$ and $\overline{\omega}$ can be considered constant. And since for all acceptable hypotheses the tidal deformation of the Earth has so small an effect on the shape of the inertia ellipsoid, the equations for G are invariant for any assumption about the interior of the Earth. We therefore observe that the above equations yield the motion of $\overline{G}$ and not of the instantaneous axis of rotation. Therefore, in the formulas for the computation of the declination, we have essentially measured from a plane $\perp \overline{G}$ → "dynamic equator." So the position of the pole also is defined as the point where $\overline{G}$ intersects the globe. The instantaneous axis moves around $\overline{G}$ on a conical surface with angle ≈0".002 with period almost 1 sidereal day. Therefore, the declination will also show a diurnal term as well as the motion of the pole, because the latitude observations are made from G. /208

We seek now the differential equations of $\overline{G}$ relative to the Earth or the differential equations of the system $OX_0Y_0Z_0$ with respect to $OX\psi Z$ or even the differential equations of OXYZ with respect to $OX_0Y_0Z_0$, rotating with angular velocity ω. We find the relations

$$\dot{X} + (1-K)\alpha^2 n\, Y = Cn\,(1-K_0)\,(-\sin\vartheta\,\cos\varphi\,\dot{\psi} + \sin\varphi\,\dot{\vartheta})$$

$$\dot{Y} - (1-K)\alpha^2 n\, X = Cn\,(1-K_0)\,(\sin\vartheta\,\sin\varphi\,\dot{\psi} + \cos\varphi\,\dot{\vartheta}) \qquad \text{thus } \overline{G}\wedge\dot{\overline{G}} = 0 \;\; (G \perp \dot{G})$$

$$\dot{Z} = 0$$

where X, Y, Z are the projections of $\overline{G}$ on $OX_0Y_0Z_0$. The integration of these relations gives the equations of motion of XYZ with respect to $OX_0Y_0Z_0$. The angle between $\overline{G}$ and $\overline{\omega}$ is of the order of 0".002. From the integration of the above relations and after substitution of

$$x = \frac{X}{Cn} \quad , \quad y = \frac{\psi}{Cn}$$

we find the formulas which yield the coordinates of the pole. This is because Cn = G and x, y = components of G:

$$G n \hat{x} = X, \quad G n \hat{y} = Y$$

From elasticity theory we derive /209

$$\frac{2n}{T} = (1-K)\alpha^2 n = (1-K)\frac{C-A}{C}\,2n$$

(we define the sidereal day as the mean time)

$\underline{A} = 304\ (1 - K) = 0.72 \rightarrow T = 433$ days,

i.e. we find the period of Chandler.

Thus elasticity theory, on the one hand, suggests the solution with superposition, and on the other hand it results only in the increase of the free period and the decrease in the values of the nutational terms. It remains now to compare the modern theory and the results obtained by observations. For this, we will examine if the forced motion takes place according to the assumption of an elastic Earth. After a brief presentation of the manner in which we obtain the constants, we will examine whether the value of, for example, N is different in theory from that in the observations. Finally, we will make a comparison with comments.

9.3. Forced Polar Motion of the Earth

9.3.1. Introduction

From the kinematics of a rigid body it is known that the motion of the axes of rotation of the body in space is always accompanied by a displacement of the axes with respect to the body. Thus, in the Earth, the nutational motion of the instantaneous axes of rotation is followed by a cyclic motion of the pole. Later, during its rotation, it seems to cause small changes in latitude, the period reaching 1 day, which are known as the terms of Oppolzer. In Chapter 1(6), the following expression was obtained for these terms, starting from the hypothesis that the Earth is an ideally elastic body.

/210

$$\Delta\varphi = -0''{,}0066 \sin S - 0''{,}0051 \sin (S - 2\mathbb{C}) - 0''{,}0022 \sin (S - 2L) - 0''{,}0010 \sin (S - 2\mathbb{C} - \Omega)$$
$$- 0''{,}0010 \sin (S - 3\mathbb{C} + \Gamma') + 0''{,}0009 \sin (S - \Omega)$$

where S is the sidereal time at the position L, $\mathbb{C}$, Ω', Γ' are the mean length of the Sun, of the Moon, the eastern point of the orbit of the Moon, and the perigee of the Moon, respectively.

We will now try to find whether the observations verify the validity of this theoretical law. The problem obviously is to search out the changes in latitude from all, or at least from some periodic causes, as shown in (31). We have already shown some difficulties arising in the interpretation of the results of analysis of the observations. We now consider them in detail.

The fact of the 14-day nutation in declination can be expressed by the formula

$$\Delta\delta = -0''.085 \sin(2\mathbb{C} - \alpha) + 0''.003 \sin(2\mathbb{C} + \alpha)$$

If the exact 14-day change in latitude is not obtained by this formula. then, terms appear in the change in latitude with arguments $(2\mathbb{C} - \alpha)$ and $(2\mathbb{C} + \alpha)$ and because in the definition of latitude according to Talcott, the stars are observed when the Sun is in the meridian, the sidereal time is always equal to the right ascension of the star at the moment and place of observation. Therefore, the arguments $(2\mathbb{C} - \alpha)$. $(2\mathbb{C} + \alpha)$ coincide. Hence, if during the change in latitude a term with argument $(2\mathbb{C} - \alpha 2\mathbb{C} + \alpha)$ is discovered and its amplitude has a different value from the theoretical one, this can be explained as an inaccuracy of the coefficient of the first term in formula (3.2), and also as an inaccuracy in the theoretical expression of the diurnal term of the Moon. This is the second term in the right-and side of (3.1).

In general, we may consider each of these explanations, but we must justify the fact that we are going to choose the one which is the most appropriate for both the theoretical research and the reduction of the observations. /211

Theory and observation could find at any instant the position of the system of the principal axes of inertia with respect to any fixed coordinate system. For this it is enough to take as examples three equations giving the dependence on time from the Euler angles, in a way in which the relative position of the two systems with the same origin is defined.

However, these equations are not appropriate, considering the rotation of the Earth. They are complex and not easy to solve. The solution becomes much simpler and descriptive if we use an auxiliary system and define its motion on the one hand with respect to the fixed system, and on the other with respect to the principal axes of inertia of the Earth. We are free to choose such an auxiliary system and it seems that there exists an arbitrariness in the explanation of the results from the analysis of the observations, considering the Moon terms in the change in latitude, due to the arbitrary system. The OZ axis of this system can be taken, for example, as the instantaneous axis of rotation or the axis of angular momentum, but if we assume that the Earth is composed of a solid (rigid) crust and a fluid core, we can take it as the axis of rotation or the axis of the angular momentum of the crust.

In each case, obviously, we will have slightly different equations of motion of the auxiliary system relative to both /212
the fixed system and the system of the principal axes of inertia of the Earth. In Fig. 7, OZ is the axis of the auxiliary system of coordinates, OZ' is the vertical, OS the direction toward any given star. We ignore the tidal change of the vertical and the proper motion of the star S, so we can assume that neither OZ' changes its direction relative to the principal axes of inertia of the Earth, nor OS its direction with respect to the fixed coordinate system of axes, for example, the system of the ecliptic at an initial time.

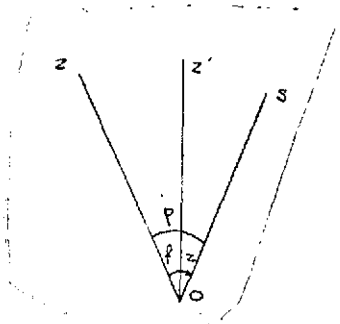


Fig. 7.

We have noted that there are some possible directions of the axis OZ. Nevertheless, all these are so close together that in each case we can take the plane of the diagram as the meridianal plane. We place

$$Z'OS = z, \quad Z'OZ = f, \quad ZOS = p \quad \Rightarrow \quad z = p + f \qquad (3.3)$$

Assume that the dependence of p and f on time is taken on the basis of some assumptions about the mechanical properties of the Earth. From the observations we can obtain the empirical expression for the change in Z. Thus it is possible to check the theory by comparison with the observations. However, the results of such a comparison will not show how much of each error we find is attributed to p(t) or to f(t).

We return now to the matter of the Moon terms in the change in latitude. Earlier, these terms were justified as an inaccuracy in the 14-day nutational term which enters the equation of motion of the OZ axis in space. (This is the /213 periodic change of the argument $2\mathbb{C}-\alpha'$ and not the tidal

change in latitude. However, as was shown (in the modern theory), the formula of nutation describes the motion of the angular momentum $\overline{G}$ which remains the same for any logical assumption about the mechanical properties of the Earth. Hence if we take the auxiliary axis OZ along the vector $\overline{G}$, its motion in each case satisfies the equations obtained by the assumtion that the Earth is rigid. Then the difference between theory and observed changes of Z can be totally attributed to the inaccuracy of the expression for the angle f(t), which in the equations determines the motion of the auxiliary system relative to the principal axes of inertia. In any case, $f = 90° - \phi$. Hence, the comparison with the observations can answer the question of whether the expression for the forced latitude change, which is obtained by considering the Earth as an ideally elastic body, corresponds to reality.

For this comparison we restrict our attention to the β' term of (31) which corresponds to the lunar diurnal term. The first term enters as partial (part) constant quantity in the value of the latitude, obtained from observation of an individual pair. The coefficients from the other terms are so small that they are not easily determined with accuracy from the observations.

9.3.2. The Lunar Diurnal Term in the Latitude Change, According to Information Given by the I.L.S. Initial Conditions and Plan of Calculation

The harmonic analysis of the I.L.S. observations in determining the amplitude and phase of the lunar diurnal term in the change in latitude requires a long calculation. In trying to abbreviate /214 it, it is necessary to put the initial conditions in as compact as possible a formula.

For this reason, we divide the 96 pairs of the I.L.S. program into 24 groups with four pairs in each. These hourly groups, in contrast to the groups of 2 hours, are used regularly in the observation work of the I.L.S., and they will be called "links," a term used by us in the description of the convergent observations of the two zenith telescopes in Poltava. The mean right ascension of the stars, which forms the links, is 0.5, 1.5, ... 23.5 hours. We also choose the complete observations of the link, which are those from which no pair was neglected. For each complete link we find the mean value of the latitude, which is the arithmetic mean from four instantaneous latitudes, obtained from observations of the pairs which belong to the link. All the links not completely observed are disregarded.

Then the initial quantities of the material from Carloforte were reduced by 18%, from Ukiah by 7%, and from Mizusawa, 31%. To compensate for this abbreviation, we are able to obtain greater homogeneity (compactness) of the given information and simplicity of computation. After this preparation, we take the residuals of the latitude which are obtained for separate links from the smoothed-out curve of the latitude oscillation. The residuals denoted by $\Delta\phi$ are subject to harmonic analysis for the lunar diurnal term. The first step is to arrange the $\Delta\phi$ values according to the phase of the argument (2☾ − α) and after that, of the argument (2☾ + α) where α is the mean right ascension of the pair constituting the link. For this, it is /215
necessary to take the mean value of twice the length of the Moon for all the mean values of the observations of the complete links. Thus, we construct auxiliary Table 25.

TABLE 25.

2☾	Date	Sidereal time	2☾	Date	Sidereal time
1	2	3	1	2	3
0^h	1913. Jan. 12.47	0.28	12^h	1913 Jan. 19.30	0.13
1	13.04	.85	13	19.86	.70
2	13.61	.42	14	20.43	.27
3	14.18	.99	15	21.00	.84
4	14.74	.56	16	21.57	.41
5	15.31	.13	17	22.14	.98
6	15.88	.70	18	22.71	.56
7	16.45	.27	19	23.28	.13
8	17.02	.84	20	23.85	.70
9	17.59	.42	21	24.42	.27
10	18.16	.99	22	24.99	.84
11	18.73	.56	23	25.56	.41
	19.30	.13		26.13	.98

In column 1 we give 2☾ for the mean points of the time intervals between instances given in column 2 in Universal time. In column 3, we give the corresponding mean Greenwich times. We do not give the complete table included on p. 78 [sic] and covering the time from 1899 to 1934.

We will illustrate the method by an example. Assume that we with to find the approximate value, two times, of the longitude of the Moon at the mean instance of observation of the fifth link at Mizusawa the night of January 18-19, 1913. The local sidereal time of the observation of this link is its right ascension, which is $4^h.5$. Because the longitude of Mizusawa is $-9^h.4$, the mean value of the observation of the pair is always $19^h.1$ of the sidereal time in Greenwich, i.e. $4.5 + (-9.4) = 19.1$.

This is equal to 0.80 for a fraction of a day. In the /216
Table 25, we find that the observation time lies in the interval between January 18.73 and January 19.30. Since the first time corresponds to 0.56 in the column for sidereal time and the second to 0.13, we find that the two times of the longitude of the Moon at the time of observation were approximately 11.

In this way we divide all the $\Delta\phi$ values into 24 groups, acccording to the argument $2☾$, expressed in hours. Moreover, for each group we compute the mean value of $\Delta\phi$ and put it in a square table, a sample of which is given in Table 26.

In the square entries of the table, we write the values of the arguments $2☾-\alpha$ and $2☾+\alpha$. After this, in order to obtain the mean values from all the $\Delta\phi$ values corresponding to equal values of the argument, it is necessary to divide the sum of the numbers in a scale by 24. So, for example, the argu- /217
ment $2☾-\alpha$ is equal to 1 hour for the $\Delta\phi$ values which we write into the square entries of the table. Similarly, we have the mean for the argument $2☾+\alpha$ by dividing by 24 the sum in the next diagonal. We use the values of the instantaneous latitude at Carloforte, Mizusawa, and Ukiah from 1899 to 1934. During this period, the program of observations changes three times at 1906.0, 1912.0 and 1912.7.

To calculate the reduction at the phenomenal position, the central office of I.L.S. used the (reducing) quantities published in the Berliner Jahrbuch [Berlin Yearbook]. The formula for the computation of these quantities changed slightly in 1916. Some small nutational terms were introduced, not previously taken into consideration. Therefore, we decided to divide the total sequence of observations from 1899 to 1934 into five cycles: (1) 1899-1906.0; (2) 1906.0-1912.0; (3) 1912.0-1916.0; (4) 1916.0-1922.7; (5) 1922-7-1935.0.

TABLE 26.

	$0^h.5$	$1^h.5$	$2^h.5$	$3^h.5$	$4^h.5$	$5^h.5$	$6^h.5$	$7^h.5$	$8^h.5$	$9^h.5$	$10^h.5$
$0^h.5$											
1.5											
2.5											
3.5											
4.5											
5.5											
6.5											
7.5											
8.5					8						
9.5											
10.5											
11.5											
12.5											
13.5											
14.5											
15.5											
16.5											
17.5											
18.5											
19.5											
20.5											
21.5											
22.5											
23.5											

First we elaborate on the observations separately for each cycle. The values of $\Delta\phi$ are obtained from the observations for each link and are divided into 24 groups, as we explained, according to the argument 2α. Then, in order to avoid the error in the declination of the pairs that we may find (which might happen) in the composition of the results of separate cycles, if the observations are not similarly distributed to the phase, we do the following:

We take the mean value for each column in Table 26. We subtract this value from the $\Delta\phi$ values of the separate square entries of the column. After that, we combine the results of the observations from the first three cycles and the next two cycles, so that we find the mean values of $\Delta\phi$ obtained separately from the I, II, III cycles and the IV and V cycles. In order to allow for differences in the numerical values of the observations, /218 we gave a weight of 1 to the results of all cycles except the Vth cycles, and a weight of 2 to the results from Carloforte and Ukiah. Thus we obtained two tables for each station, so that we have six tables in all. We give one of them as an example, the 1900-1915 cycle for Carloforte (Table 27).

Under the heading "diagonal sum," the sum of the numbers in /220
the diagonal is given in the first column (descending), going from the upper left corner of the table to the lower right corner. In the "ascending" column, the sum of the numbers in the diagonal is given from the lower left to the upper right corner. Under the heading "phase" we give the values of the phases of the arguments $2☾ - \alpha$ and $2☾ + \alpha$ corresponding to these sums. The $\Delta\phi$ values in Table 27 are given in 0".001. The mean values, which are these sums divided by 24, are given in Table 28 in the column with the heading $\Delta\phi'm$ for the 1900-1915 cycle and $\Delta\phi m$ for the 1916-1934 cycle, expressed in 0".001.

9.3.3. Corrections of the Nutational Term, Argument $2☾ - \Omega$ /222

Preliminary analysis of the values of $\Delta\phi m$ showed that the expressions for the lunar diurnal term obtained from observations in the years 1899-1915 and 1916-1934 differ systematically. In an effort to explain this disagreement, we gave particular attention to the fact that from 1916 on some new nutational terms entered the formula, giving the quantities A' and B' in the <u>Berliner Jahrbuch</u>. Among them, the following exist

$$\Delta\delta = -0''.014 \sin(2☾ - \Omega)\cos\alpha + 0''.018 \cos(2☾ - \Omega)\sin\alpha \qquad (3.4)$$

Since earlier than 1916 this term was not taken into account in the analysis of change in latitude, there appeared a fictitious term with argument $2☾ - \Omega$. The introduction of the correction due to this nutational term in the initial information would require long additional calculations. However, we can avoid making these calculations if we apply the following method of approximation, allowing for the introduction of the correction according to the mean values $\Delta\phi m$. These corrections are denoted by $\Delta\delta m$.

We transform expression (3.4) as follows

$$\Delta\delta = \nu \cos\Omega + V \sin\Omega \qquad (3.5)$$

TABLE 27.

2☾ \ α	0ʰ.5	1ʰ.5	2ʰ.5	3ʰ.5	4ʰ.5	5ʰ.5	6ʰ.5	7ʰ.5	8ʰ.5	9ʰ.5	10ʰ.5	11ʰ.5	12ʰ.5	13ʰ.5	14ʰ.5	15ʰ.5	16ʰ.5	17ʰ.5	18ʰ.5	19ʰ.5	20ʰ.5	21ʰ.5	22ʰ.5	23ʰ.5	Diagonal Sum Descending	Diagonal Sum Ascending	Phase
0ʰ.5	-10	-20	-42	+32	-52	+23	+46	+25	+24	-34	-12	+5	-5	+43	+40	+7	+33	-7	+5	+15	+15	+27	+21	+29	-65	-20	0ʰ
1.5	+2	+49	-4	-56	-10	-16	-56	-37	-7	+47	-8	+23	-44	+30	+9	+27	+7	-1	+4	-7	+4	+1	+4	+4	-9	+125	1
2.5	+40	+35	+18	-4	-86	-20	-7	+3	-19	+5	+14	-16	+30	-32	-43	-12	+27	+16	+7	+22	-15	+12	+2	+24	+79	-61	2
3.5	+6	+11	+23	-6	-43	-42	-8	-5	-36	-40	0	+26	-4	+7	-3	+7	-12	+31	+27	-1	-1	+29	+22	+19	+69	+118	3
4.5	+17	+1	+8	-43	-1	-16	-63	-5	-25	-16	-63	-31	-20	+1	-8	+16	-6	-34	-16	+3	+5	+29	+18	+29	-2	+227	4
5.5	-8	-30	+45	+39	+43	-34	+28	-15	+14	-11	-35	-73	-11	-62	+40	-2	+4	+16	-9	+4	-18	-5	+12	-23	+291	+65	5
6.5	+5	+30	+4	+18	+42	-10	-46	-36	-22	-41	-38	-14	-14	-10	+32	+7	+16	+13	-12	0	-29	-13	-7	-20	+290	+52	6
7.5	+15	+17	+32	-4	+13	-25	-3	-18	-48	-9	-27	-49	+2	-43	-10	+13	+3	+7	+2	-3	-15	-10	+23	+31	+175	+40	7
8.5	+17	+24	+70	+16	-13	-9	-37	-12	-30	-65	-31	+11	+8	-21	+1	-38	-23	-24	-7	-12	-7	-7	-2	+6	+413	+74	8
9.5	+7	+5	+24	+54	+24	-14	-14	-6	+19	+4	+39	+29	-26	+15	+3	+31	+24	-14	+25	+6	-8	-15	+12	+11	+226	-67	9
10.5	-12	-8	+17	+22	-14	+14	+14	-8	+6	+36	+9	+2	-3	+43	-12	-9	-10	+15	+1	+1	-15	-12	-27	-1	+270	+61	10
11.5	+3	-4	-10	-15	-1	+27	+9	-30	-26	-6	+14	-31	-12	+7	-15	+15	+6	-7	-42	-1	+12	-12	-21	-34	+71	+56	11
12.5	+1	-8	-44	-0	+35	+8	+12	0	+37	-1	+14	+1	-13	-39	-6	+21	+8	+39	+17	+18	-8	+20	-35	-7	+13	-77	12
13.5	-19	-34	+26	-19	+18	+13	+49	+18	+33	+3	-19	+8	-6	+1	-7	-38	+4	-32	+6	-15	-1	+20	-64	-8	+23	-154	13
14.5	+21	+8	+9	+28	-6	+4	+52	-1	-5	-26	+5	-47	-28	-34	-19	-43	+10	+12	-10	+6	-11	+13	+10	-5	-36	-128	14
15.5	-1	+7	+42	-2	-17	+26	-49	-2	-18	+10	+1	+10	+14	-4	+17	-5	-18	-17	-12	+17	+5	-39	-46	-23	-23	-133	15
16.5	-10	+64	-40	-55	+13	+8	-6	+44	+32	+13	+14	+22	+21	+24	-4	+8	-14	-18	+3	-25	-16	-5	-7	+1	-18	-126	16
17.5	-59	+1	-20	+7	+29	-11	-10	-36	+15	+16	+40	-18	+41	+9	-13	-37	-25	+3	-20	-12	-5	-23	+19	-21	-198	-70	17
18.5	-8	-33	-44	-16	-42	-27	+37	0	+45	+12	+42	-53	+22	-16	-1	+18	-2	-18	+5	-7	-8	-17	-8	-32	-292	+11	18
19.5	-14	-11	0	+6	+3	+36	+63	+27	-5	+120	+31	+62	+23	-11	+2	-25	-37	-14	-22	-9	-2	+3	+19	-12	-147	-184	19
20.5	+21	+8	-36	-5	+5	-28	+16	+53	-3	+31	+43	+61	+6	+25	+8	+1	+25	-1	+3	+6	-11	+11	-10	-9	-242	+3	20
21.5	-60	-21	-46	+8	+4	-18	+22	+22	+17	-4	-10	+26	+3	+37	-12	+12	+8	-12	+14	+21	+32	+24	-3	-15	-176	+196	21
22.5	-8	-3	+20	-5	+9	+21	+4	+3	-7	+36	+33	+28	+22	+37	+3	+10	+50	+40	+15	+5	+37	+11	+37	+24	-450	+21	22
23.5	+36	-19	-49	-9	+22	+77	-31	+2	-4	-2	-25	+38	+23	-6	+3	+21	-32	+20	-14	-20	+31	-12	+30	+32	-265	-27	23

TABLE 28.

Phase	Carloforte $\Delta\varphi'_m$	Carloforte $\Delta\varphi_m$	Carloforte O-C	Mizusawa $\Delta\varphi'_m$	Mizusawa $\Delta\varphi_m$	Mizusawa O-C	Ukiah $\Delta\varphi'_m$	Ukiah $\Delta\varphi_m$	Ukiah O-C	Carloforte $\Delta\varphi_m$	Carloforte O-C	Mizusawa $\Delta\varphi_m$	Mizusawa O-C	Ukiah $\Delta\varphi_m$	Ukiah O-C
	2ℭ − α (1900-1915)									2ℭ − α (1916-1934)					
0^h	− 3	− 1	+ 2	− 4	− 2	+ 3	− 2	0	+ 3	+ 7	+ 9	− 6	− 6	− 9	− 8
1	0	+ 2	+ 3	+ 3	+ 5	+ 3	+10	+12	+12	+ 2	+ 1	−20	−21	+ 2	0
2	+ 3	+ 5	+ 3	− 7	− 5	− 4	− 4	− 2	− 5	+ 2	− 1	+ 5	+ 3	+ 4	− 1
3	− 3	− 1	− 6	− 4	− 3	− 4	− 3	− 1	− 7	+10	+ 4	+14	+11	+11	+ 4
4	0	+ 1	− 6	+10	+11	+ 9	+ 4	+ 6	− 2	+ 5	− 3	+ 6	+ 2	+11	+ 2
5	+12	+13	+ 4	+ 2	+ 2	− 2	+ 5	+ 6	− 3	+ 1	− 8	+14	+10	+ 2	− 8
6	+12	+13	+ 3	+ 2	+ 2	− 4	+10	+10	0	+ 9	− 1	+ 7	+ 3	+13	+ 3
7	+ 7	+ 7	− 4	+ 3	+ 2	− 5	+ 4	+ 5	− 6	+10	0	+ 4	0	+ 6	− 4
8	+17	+17	+ 6	+ 1	0	− 7	+ 9	+ 9	− 1	+ 8	− 2	− 3	− 6	+13	+ 4
9	+ 9	+ 8	− 2	+ 2	0	− 7	+32	+32	+23	+16	+ 7	− 9	−12	+ 1	− 7
10	+11	+10	+ 2	+22	+21	+15	+ 5	+ 4	− 4	+ 8	0	− 6	− 8	+12	+ 6
11	+ 3	+ 1	− 5	+ 9	+ 7	+ 1	+ 4	+ 2	− 3	− 2	− 7	+ 1	0	+ 9	+ 6
12	0	− 2	− 5	+20	+18	+13	+13	+11	+ 6	+ 2	0	+13	+13	+ 1	− 2
13	+ 1	− 1	− 2	− 2	− 4	− 7	− 8	−11	−11	− 2	− 1	0	0	+ 1	− 3
14	− 2	− 4	− 2	+ 2	0	− 1	+ 3	+ 1	+ 4	− 1	+ 2	+ 5	+ 6	−12	− 7
15	− 1	− 3	+ 2	+ 1	0	+ 1	−11	−13	− 7	− 2	+ 4	−20	−18	− 3	+ 4
16	− 1	− 2	+ 5	− 9	−10	− 8	− 2	− 4	+ 4	−14	− 6	− 4	− 1	− 7	+ 2
17	− 8	− 9	0	− 8	− 8	− 4	− 7	− 8	+ 1	+ 3	+12	+ 4	+ 8	− 9	+ 1
18	−12	−13	− 3	−16	−16	−10	− 7	− 7	+ 3	−10	0	−14	−10	−25	−15
19	− 6	− 6	+ 5	+ 6	+ 7	−14	−14	−14	− 3	−21	−11	− 6	− 2	− 4	+ 6
20	−10	−10	+ 1	− 3	− 2	+ 5	− 3	− 3	+ 7	− 9	+ 1	+10	+13	−10	− 1
21	− 7	− 6	+ 4	−10	− 9	− 2	− 6	− 5	+ 4	−17	− 8	+10	+13	+ 2	+10
22	−19	−17	− 9	− 8	− 6	+ 1	−21	−20	−12	− 1	+ 7	− 8	− 6	0	+ 6
23	−11	− 9	− 3	−16	−14	− 8	+ 1	+ 3	+ 8	− 7	− 2	− 5	− 4	− 2	+ 1

Phase	Carloforte $\Delta\varphi_m$	Carloforte O-C	Mizusawa $\Delta\varphi_m$	Mizusawa O-C	Ukiah $\Delta\varphi_m$	Ukiah O-C	Carloforte $\Delta\varphi_m$	Carloforte O-C	Mizusawa $\Delta\varphi_m$	Mizusawa O-C	Ukiah $\Delta\varphi_m$	Ukiah O-C
	2ℭ + α (1900-1915)						2ℭ + α (1916-1934)					
0^h	− 1	− 4	− 6	− 5	+ 1	+ 2	0	0	− 8	− 4	+ 1	− 2
1	− 1	− 3	+ 8	+ 9	− 5	− 3	+ 8	+ 9	−11	− 6	+ 5	+ 5
2	+ 1	0	−13	−13	+ 5	+ 8	+ 5	+ 7	−26	−21	− 2	− 3
3	+ 8	+ 8	0	− 1	0	+ 3	− 4	− 2	0	+ 5	− 4	− 4
4	0	+ 2	0	− 1	−11	− 7	−12	− 9	0	+ 4	− 2	− 1
5	− 7	− 5	− 3	− 5	+ 2	+ 6	−12	− 9	+ 2	+ 5	+ 3	+ 5
6	0	+ 3	+22	+20	− 8	− 4	0	+ 3	+ 5	+ 7	− 1	+ 2
7	− 3	+ 1	+ 2	− 1	− 1	+ 2	+ 3	+ 6	+ 1	+ 2	+ 7	+11
8	− 5	− 1	+ 3	0	+ 4	+ 6	− 5	− 2	+ 4	+ 4	− 6	− 2
9	− 6	− 2	− 4	− 6	−10	− 8	− 1	+ 1	− 2	− 3	− 6	− 2
10	− 5	− 1	+ 7	+ 5	0	+ 1	+ 1	+ 3	+ 4	+ 1	− 7	− 3
11	− 6	− 3	+ 5	+ 3	+ 6	+ 6	− 2	− 1	+ 1	− 3	− 5	− 1
12	− 3	0	−10	−11	− 7	+ 6	− 1	− 1	− 8	−12	− 6	− 3
13	+ 2	+ 4	− 8	− 9	− 4	6	+ 5	+ 4	− 3	− 8	− 9	− 7
14	+ 2	+ 4	− 2	− 2	− 2	− 5	+ 2	0	+ 4	− 1	+ 9	+10
15	− 3	3	+ 9	+10	+10	+ 7	− 2	− 4	+12	+ 7	− 5	− 5
16	+ 2	+ 1	5	4	2	6	− 1	− 4	+19	+15	+ 1	5
17	+ 2	0	+ 6	+ 7	+11	+ 7	+ 7	+ 4	+ 9	+ 6	+ 4	+ 0
18	+ 2	− 1	− 3	− 1	+ 8	+ 4	+ 6	+ 3	− 8	−10	+12	+ 2
19	+ 2	− 1	+ 2	+ 6	+ 4	− 7	+ 1	− 2	−12	−13	− 1	− 9
20	+ 9	+ 5	− 3	0	− 2	− 4	+ 5	+ 2	− 6	− 6	+ 9	+ 5
21	+ 5	+ 1	− 7	− 5	+ 6	+ 4	+ 1	− 1	+ 9	+10	− 1	− 5
22	− 3	− 7	− 4	− 2	+ 8	+ 7	− 8	−10	+11	+14	+ 9	+ 4
23	+ 5	+ 2	0	+ 2	− 6	− 6	+ 4	+ 3	0	+ 4	0	− 4

where

$$v = -0\rlap{.}''016 \sin(2☾ - \alpha) - 0\rlap{.}''002 \sin(2☾ + \alpha)$$

$$V = 0\rlap{.}''016 \cos(2☾ - \alpha) + 0\rlap{.}''002 \cos(2☾ + \alpha) \qquad (3.6)$$

Starting from (3.5), we obtain the expression

$$\Delta\delta_m = v\frac{1}{n}\Sigma\cos☊ + V\frac{1}{n}\Sigma\sin☊$$

where n is the number of values of $\Delta\phi$ from which the mean value $\Delta\phi m$ is formed for a given group. Since the values are distributed over phases Ω almost uniformly, we can substitute the sum by integrating the previous formula which becomes

$$\frac{1}{n}\Sigma\cos☊ = \frac{1}{☊_1 - ☊_0}\int_{☊_0}^{☊_1}\cos☊ \, d☊ \qquad /223$$

$$\frac{1}{n}\Sigma\sin☊ = \frac{1}{☊_1 - ☊_0}\int_{☊_0}^{☊_1}\sin☊ \, d☊$$

where Ω_1 and Ω_0 are the values of the longitude of the position of the Moon at the beginning and end of the observation cycle. In computing these values, we took 1899.8 for the beginning of observations in Carloforte and Ukiah and 1900.0 in Mizusawa. 1916.0 was taken as the end of the cycle for all the above stations. After the substitution of the arithmetic values, we finally obtain, for the correction of the nutational term with argument $2☾ - ☊$ for: Carloforte and Ukiah

$$\Delta\delta_m = 0\rlap{.}''0006 \sin(2☾ - \alpha) + 0\rlap{.}''002 \cos(2☾ - \alpha)$$

and Mizusawa

$$\Delta\delta_m = 0\rlap{.}''0021 \cos(2☾ - \alpha)$$

The corrected $\Delta\phi'm$ values are in Table 28 under the heading $\Delta\phi m$.

9.3.4. Results

We can classify the $\Delta\phi m$ values of Table 28 as follows:

Cycle 1900-1915:

Carlo-forte $\rightarrow$ $0''.0108 \sin(2\text{☾} - \alpha - 18°) + 0''.0040 \sin(2\text{☾} + \alpha + 39°)$

$\pm 11 \qquad \pm 6 \qquad \pm 14$

Mizu-sawa $\rightarrow$ $0''.0079 \sin(2\text{☾} - \alpha - 39°) + 0''.0024 \sin(2\text{☾} + \alpha - 158°)$

$\pm 22 \qquad \pm 17 \qquad \pm 21 \qquad \pm 46$

Ukiah $\rightarrow$ $0''.0104 \sin(2\text{☾} - \alpha - 15°) + 0''.0035 \sin(2\text{☾} + \alpha - 16°)$

$\pm 23 \qquad \pm 12 \qquad \pm 17 \qquad \pm 27$

Cycle 1916-1934:

Carlo-forte $\rightarrow$ $0''.0105 \sin(2\text{☾} - \alpha - 11°) + 0''.0032 \sin(2\text{☾} + \alpha - 2°)$

$\pm 16 \qquad \pm 9 \qquad \pm 14 \qquad \pm 24$

Mizu-sawa $\rightarrow$ $0''.0041 \sin(2\text{☾} - \alpha + 5°) + 0''.0046 \sin(2\text{☾} + \alpha - 63°)$

$\pm 20 \qquad \pm 39 \qquad \pm 26 \qquad \pm 30$

Z224

Ukiah $\rightarrow$ $0''.0104 \sin(2\text{☾} - \alpha - 3°) + 0''.0043 (2\text{☾} + \alpha + 46°)$

$\pm 17 \qquad \pm 9 \qquad \pm 15 \qquad \pm 18$

The differences between the observed values and those which are computed from these formulas are expressed in multiples of 0".001 and are given in Table 28 in the O-C column. The numbers in these columns are used for the determination of the mean errors of the values found for the amplitude and initial phase of the lunar terms.

The expression for the lunar diurnal term can be written as follows:

$$\Delta\varphi_m = M_1 \sin(2\mathbb{C} - \alpha) + N_1 \cos(2\mathbb{C} + \alpha) + M_2 \sin(2\mathbb{C} + \alpha) + N_2 \cos(2\mathbb{C} + \alpha) \qquad (3.9)$$

which is more appropriate for the combination of the results of some sequences of observation. A summary of the values of the coefficients M_1, N_1, M_2, N_2 is given in Table 29. An important part of the calculations for the lunar diurnal term was carried out by Miss Vertushenko, and the preliminary results of the calculations were published simultaneously with ours.

9.3.5. Results of the Research of Other Authors

The study of the lunar terms in latitude change was, until recently, restricted to the derivation of the major lunar half-diurnal tidal term, with argument $2\mathbb{C} - 2S$, where S is the local sidereal time. The first arguments for the discovery of the lunar diurnal term were almost simultaneously and independently presented by Sekiguchi, Morgan and Popov. They first explained this term, as Fedorov also did, as an inaccuracy in the 14-day term. The fact that we now present is a different explanation which does not prevent us from using the results in the general /225
study which follows. In the determination of the coefficients of the lunar terms, Sekiguchi did the following. He generally reconsidered the results (abstractions) of the precession and nutation formula, and he obtained computed values for the coefficients which were different, for some terms, from those that Oppolver found. Thus, for the 14-day term, according to Oppolver we have

$$\Delta\delta = 0''.088 \sin\alpha \cos 2\mathbb{C} - 0''.0081 \cos\alpha \sin 2\mathbb{C}$$

while Sekiguchi's result is given by the formula

$$\Delta\delta = -0''.091 \sin(2\mathbb{C} - \alpha) + 0''.003 \sin(2\mathbb{C} + \alpha)$$

$$\Delta\delta = 0''.095 \sin\alpha \cos 2\mathbb{C} - 0''.088 \cos\alpha \sin 2\mathbb{C}$$

or in the form (3.2). If the last expression is correct, and we take formula (3.2) in the reduction to the phenomenal position, a term appears which is

$$0\overset{''}{.}006 \sin (2\zeta - \alpha)$$

To this we must add the lunar diurnal term of Oppolver, coinciding by chance with the last expression for a rigid Earth, so that we have as the sum

$$\Delta\varphi = 0\overset{''}{.}012 \sin (2\zeta - \alpha)$$

If such a term was discovered in the change in amplitude, according to Sekiguchi, this could be taken as a proof that his theory for nutation is better than Oppolver's. However, we do not have such a valid conclusion, because the difference between the formulas is explained by an error made by Sekiguchi. This error was shown by Woolard (see introduction). As initial material for the determination of the amplitude of the lunar sidereal term, Sekiguchi used the data from Carloforte, Ukiah /226 and Mizusawa from 1922 to 1934.

The above observations concern only the interpretation of the results. They agree, as a whole, satisfactorily with ours. We restrict ourselves to this remark, and we do not put the results of Sekiguchi in the general summary because (1) the I.L.S. observations for 1922-34 deal with more material than we used, and (2) the method of reduction he used is subject to some objections.

Morgan first announced the results of his study on the 14-day term at the seminar on astronomic constants in Paris (March, 1950). His results were based on the interpretation of the 4-yearly sequences of observations with the PZT in Washington. After this, complete material was worked out. 13-yearly sequences were used, but they were not taken one after the other, but with some interruptions, so that the total sequence covers an interval of 19 years.

Morgan presented the lunar diurnal term in the formula

$$\varphi_0 - \varphi = (\cos 2\zeta \cdot \sin \alpha - 0{,}92 \sin 2\zeta \cos \alpha)\, \Delta k$$

and he determined the coefficient ΔK from observations. We rearrange this in the formula

$$\varphi - \varphi_0 = 0{,}96\,\Delta K \sin(2\text{☾} - \alpha) + 0{,}04\,\Delta K \sin(2\text{☾} + \alpha)$$

and we note that this coefficient practically coincides with the coefficient M_1 in (3.9). Morgan found that $\Delta K = 0''.0067 \pm \pm 0''.0020$, which coincides with the theoretical value of the coefficient of the corresponding term in Oppolver's formula. On the basis of these, Morgan concluded that the lunar diurnal term in the change in amplitude at Washington is totally due to /227
forced nutation of the pole and thus the coefficient of the 14-day term does not require correction.

From Morgan's work we can take only the value of M_1. The results are given in such a form that it is not possible to calculate from them the coefficients for other terms in expression (3.9). From the analysis of the observations of two bright zenith stars in Poltava, Popov obtained a conclusion opposite to that of Morgan. He found clear proof in these observations of the 14-day term, and explained this as an inaccuracy in the value obtained for the amplitude of the 14-day nutational term. From 780 observations of Alpha Perseus and 925 of Eta Ursae Majoris, he found that

$$\Delta\varphi_\alpha = 0''.028 \cos(2\text{☾} + 232°)$$
$$\Delta\varphi_\eta = 0''.034 \cos(2\text{☾} + 99°)$$

If we put these expressions in the formula in which we have the lunar diurnal term, we obtain

$$\Delta\varphi_\alpha = 0''.028 \sin(2\text{☾} - \alpha + 12°)$$
$$\Delta\varphi_\eta = 0{,}034 \sin(2\text{☾} - \alpha - 55°)$$

As we see, the amplitudes of these terms are approximately taken by Popov as two times greater than those taken earlier. It is certainly necessary to take into account the fact that his results include the main lunar half-diurnal tidal term also, but it does not seem possible to attribute the difference to it. Nor does it seem possible to explain it as something due to the untrustworthiness of Popov's results, because the separate reduc- /228
tion from three sequences of observations gave expressions for

the lunar diurnal term with enough (common) agreement among them. Thus the matter of the cases of irregularly great values of the amplitudes of these terms, obtained from the observations of the bright zenith stars in Poltava, is still open until more general material is collected and reduced. (A reduction has already been made by Popov, and the new result is in good agreement with (3.13).) For the time being, in relation to the relatively small numbers of observations, the results of Popov are not included in the general recapitulation.

Attempts have also been made to find a lunar diurnal term in observations with two zenith telescopes in a common program which was introduced in Poltava in 1949 and still continues. Matveyev used as initial material the differences between the values obtained for the latitude from morning and night observations for the first 3 years. As a mean value for the two instruments, he found:

$$\Delta\varphi = 0''.018 \sin (2\mathbb{C} - \alpha - 29)$$
$$\pm 4 \qquad\qquad \pm 13$$

With the same observations but for a greater period, Filippov, using a method similar to the one we used in the analysis of the I.L.S. observations, found the following value for the term considered:

$$\Delta\varphi = 0''.0126 \cdot \sin (2\mathbb{C} - \alpha + 8^\circ)$$
$$\pm 26 \qquad\qquad \pm 12$$

Finally, for one more time studying the lunar diurnal term for the difference "evening minus morning" and making observations for 6 years, we found

$$\Delta\varphi = 0''.0090 \cdot \sin (2\mathbb{C} - \alpha + 21^\circ)$$ /229
$$\pm 30 \qquad\qquad \pm 19$$

In order to sum up all the results, we remark that, in the observations at Poltava in the 1949 program, a lunar diurnal term does indeed exist with amplitude exceeding the corresponding term in Oppolver's formula. Certainly the number of observations from which these results were obtained is still not satisfactory enough to be suitable to be included in the general summary.

We give the expression for the lunar diurnal term obtained by Orlov from observations with the zenith telescope at Pulkovo from 1915 to 1928.

$$\Delta\varphi = \underset{\pm 16}{0''.0127} \sin(2\text{☾} - \alpha - \underset{\pm 7}{3°}) + \underset{\pm 22}{0''.0027} \sin(2\text{☾} + \alpha - \underset{\pm 47}{56°}) \qquad (3.11)$$

9.3.6. Corrections in Vertical Change Due to the Tide

Besides forced polar motion, the tidal changes in the vertical can be a case of small period change in latitude. This fact must be excluded, which can only be done by a calculation based on tidal theory. For a rigid Earth, we have the following expression for the tidal change of latitude:

$$\Delta\varphi = -\frac{1}{\alpha g}\frac{\partial v}{\partial\varphi}$$

where α is the radius of the Earth, g is the gravitational acceleration, and v is the potential of the force due to the tide. Next is the sum of the periodic terms from which we use only the one with the same period as the lunar diurnal term we are interested in, which is O_1. Substituting in the previous /230
equation the following expression for this term:

$$-9,86 \cdot \sin 2\varphi \cdot \sin(2\text{☾} - \alpha)$$

then

$$\rightarrow \Delta\varphi = 0''.0065 \cdot \cos 2\varphi \cdot \sin(2\text{☾} - \alpha)$$

Then, in order to obtain the value of $\Delta\phi$ for an elastically deforming Earth, it is enough to multiply the expression by some coefficients d depending on the mechanical properties of the Earth, which we can roughly calculate. It seems that d is slightly greater than 1, and we will take it as 1.1. Then, from (3.12), we find the following expression for the tidal change in latitude (the O_2 term in the note about the tide):

	φ	$\Delta\varphi$
Carloforte, Mizusawa, Ukiah	39° 8′	$0''.0015 \cdot \sin(2\text{☾} - \alpha)$
Washington	38° 55′	$0''.0016 \cdot \sin(2\text{☾} - \alpha)$
Pulkovo	39° 46′	$-0''.0035 \cdot \sin(2\text{☾} - \alpha)$

These values of $\Delta\phi$ must be subtracted from the expressions of the lunar diurnal term in latitude change, which we found earlier from the analysis of the results.

9.3.7. Final Expression of the Lunar Diurnal Term in Change in Latitude

For the most probable values of the coefficients M_1, N_1, M_2, N_2 in (3.9), we have the results which we obtained from the I.L.S. observatories and also those of Morgan and Orlov. These results are given in Table 29. Column 4 has the values of M_1 found by observation, but the values of column 5 are corrected for the influence of tidal change of the vertical. The computation was carried out in two ways. /231

In the first, the weights of the individual values were taken proportional to the number of observations of the corresponding sequences; they are denoted by P.

In the second, the weights were taken inversely proportional to the constant errors (mean squares of most probable values). Since the errors in the values of M_1 and N_1 are always taken equal for these, we give a simple (separate) weight p_1. Similarly for M_2 and N_2, we give a weight p_2. The first method yields

$$M_1 = 0''.0086 \pm 0''.0014 \qquad N_1 = -0''.0019 \pm 0''.0006$$

$$M_2 = 0''.0021 \pm 0.0007 \qquad N_2 = 0''.0001 \pm 0''.0010$$

The errors were determined from the deviations in separate values of the coefficients, from the "center-weight mean values" given above.

The second method gives

$$M_1 = 0''.0091 \pm 0''.0006 \qquad N_1 = -0''.0023 \pm 0''.0006$$

$$M_2 = 0''.0026 \pm 0''.0006 \qquad N_2 = 0''.0007 \pm 0.0006$$

Here the mean error was computed from the independent errors of measurement unequal with respect to accuracy (least squares method). Corresponding to the two computational methods, we find the following expressions for the lunar diurnal term:

$$\Delta\varphi = 0''.0088 \cdot \sin\,(2☾ - \alpha - 12°) + 0''.0021 \cdot \sin\,(2☾ - \alpha + 3°)$$
$$\pm 13 \qquad\qquad \pm 8 \qquad\qquad \pm 70 \qquad\qquad \pm 19$$
$$\Delta\varphi = 0''.0094 \cdot \sin\,(2☾ - \alpha - 15°) + 0''.0027 \cdot \sin\,(2☾ + \alpha + 15°)$$
$$\pm 6 \qquad\qquad \pm 4 \qquad\qquad \pm 6 \qquad\qquad \pm 13$$

After the re-establishment of the right ascension α of the observed pair by the local sidereal time S, we finally obtain

$$\Delta\varphi = -0''.009 \cdot \sin\,(S - 2☾ + 14°) + 0''.002 \cdot \sin\,(S + 2☾) \qquad (3.13)$$

Then the equations for the lunar diurnal forced motion of the pole can be written /232

$$\begin{aligned} x &= -0''.009 \; \sin\,(S_0 - 2☾ + 14°) + 0''.002 \cdot \sin\,(S_0 + 2☾) \\ y &= -0''.009 \cdot \cos\,(S_0 - 2☾ + 14°) + 0''.002 \cdot \cos\,(S_0 + 2☾) \end{aligned} \qquad (3.14)$$

where S is the sidereal time at Greenwich.

In theory, we obtained these equations of motion assuming that the Earth is an elastic body. They also include terms with arguments $S_0 - 2☾$ and $S_0 + 2☾$, representing cyclic motions with periods 1.079 and 0.932 days. In any case, the radius of the polar orbit in the second motion does not reach 0".0006, so that essentially the lunar terms in (1.38) give simply a motion of the pole on a circle of radius 0".005 and a period 1.079 days.

Equations (3.14) obtained by analyzing a great number of observations show that in reality the forced wobble of the pole that we considered takes place in a different manner. First, the motion with period 1.079 sidereal days takes place on a circle, but the radius has two times the theoretical value, close to 0".009. Second, all the observation sequences we have considered, except Mizusawa 1916-1934, give negative values of N_1. This suggests the fact that the initial phase of the motion is not zero, as is assumed in theory.

TABLE 29.

Observatory	Cycle of observations	Number of observations in thousands	M_1	M'_1	N_1	ε_1	M_2	N_2	ε	p	p_1	p_2
Carloforte	1900 — 1915	36	+0".0103	+0".0088	−0".0034	+0".0011	+0".0031	+0".0025	±0".0010	4	8	10
»	1916 — 1934	31	+ .0103	+ .0088	− .0021	.0016	+ .0032	− .0001	.0015	3	4	5
Mizisawa	1900 — 1915	21	+ .0057	+ .0042	− .0047	.0022	− .0022	− .0009	.0021	2	2	2
»	1916 — 1934	23	+ .0040	+ .0025	+ .0003	.0028	+ .0021	− .0041	.0026	2	1	2
Ukiah	1900 — 1915	28	+ .0104	+ .0089	− .0027	.0023	+ .0034	− .0010	.0017	3	2	3
»	1916 — 1934	37	+ .0104	+ .0089	− .0005	.0017	+ .0030	+ .0031	.0015	4	3	4
Washington	1931 — 1951	28	+ .0067	+ .0051	—	.0020	—	—	—	3	2	—
Pulkovo	1915 — 1928	28	+ .0127	+ .0162	− .0007	.0016	+ .0015	+ .0022	.0022	3	4	2

/233

Finally, the observations show a second cyclic polar motion of period 0.932 sidereal days, which is impossible if the orbit of the pole in this motion has the theoretical value 0".0006.

9.4. Values of Constants /234

The theoretical values are obtained as follows. If α is the mean distance of the Moon, the gravitational attraction of the Moon on the Earth causes a monthly motion of the Earth with radius $K(\alpha/1 + K)$. This affects the direction of a neighboring planet, especially of Venus, close to the opposition, as it is seen from the Earth. The proportionality of the distances of the planets is (considered) known, and the result is a monthly displacement of the Sun, known as "lunar inequality." So if the distance of the Sun is known, the absolute value of the monthly displacement of the Earth is known, and thus K is known because α is known. Thus we find the mass of the Moon.

The parallax of the Sun has been optically determined many times. The most recent and perhaps most accurate definition was given by Spencer Jones. If we can measure the perturbations of other planets due to the Earth and the Moon together, this will give the ratio (E+M)/S.

But we also have $f\left(\frac{S+M+E}{\alpha'^3}\right) = n'^2$ where α' = distance of the Sun and $f\,\frac{E+M}{\alpha^3} = n^2$ separately for small known corrections. These equations determine the α/α' and thus the distance of the Sun. Unfortunately, the best determination of these sequences, according to E. Rabe, seems to be inconsistent with the one by Spencer Jones, and it has a very small uncertainty.

Systematic errors have not been discovered in any determination, but analysis supports the one by Rabe.

The ratio of the precession is of the form

$$\frac{C-A}{C}\left(a+\beta\frac{K}{1+K}\right) \tag{1}$$

where a, b are known quantities. Hence, if K is known, we can find the (C-A)/C, the dynamic ellipticity, already called precessional constant. The main nutation for a rigid body has obliquity amplitude /235

$$N=\mu\frac{K}{1+K}\;\frac{C-A}{C} \tag{2}$$

Until 1901, (1) and (2) were made use of as the pair of equations defining n and (C-A)/C. Hink determined the lunar inequality from observation of Venus in "opposition," and he found an apparently much more accurate determination of K. Thus the use of the observed nutational constant for this purpose was considered invalid (was substituted). Nevertheless, the values observed were systematically smaller than those derived from the lunar inequality and the ratio of the nutation. Jackson first insisted that the difference was genuine.

For a rigid body, the coefficient would be N = 9".2272 or 9".2242, according to the solar precession adopted by Spencer Jones and Rabe. The effect of elasticity on the main nutation is the reason why it seems smaller. The dynamics of a rigid shell filled with a fluid were studied by Kelvin, Hough, Greenhill, and Poincaré. Poincaré gave two different methods. One was reproduced in hydrodynamics by Lamp, but the other is better, because of the generalization to an elastic shell and a nonuniform nucleus. Bonti and Lyttleton directed the attention of Harald Jeffreys to the paper by Lamb, so that he found, using Bullen's value for the moment of inertia of the nucleus, that a fluid nucleus could reduce the amplitude by 1/150.

The correction for the elasticity of the shell reduced the result, but only when Takenchi gave a complete solution for an elastic shell with properties found from seismological information, this showed that the solution for the Earth itself /236 is possible. The most appropriate method was to refer the Earth to a system rotating with the new angular velocity, introducing appropriate functions with additional displacements and to use the coefficients as Lagrange coordinates.

For a rigid Earth, the relation (ratio) of the amplitude of θ and ψ (obliquity, length) is constant. In all the analyses of observations before Fedorov, the two components were assumed to be the ratio for a rigid body. But the theoretical ratio is slightly affected by the correction due to elasticity, and affected much more by the correction due to the fluidity of the Earth's core. This arises as follows.

For any constituent material of the Earth, the precession remains the same. So if $\bar{\sigma}$, the wobble velocity referred to the axes of inertia, is zero, the constituent material does not change the forced motion, and for small variations of $\bar{\sigma}$, the changes will be proportional to $\bar{\sigma}$. Now, a nutation takes place on an ellipse, and it can be considered as the result of two cyclic motions with equal and opposite velocities. If one of them has its amplitude decreasing by some changes in the assumed constituent materials of the Earth, the other will respectively increase, and the result will be a change in the relation of the axes of the ellipse. Hence, we especially consider impossible the decrease in the theoretical amplitude of the main nutation in the obliquity without a decrease in the longitude, of a greater portion.

In other times, it was considered that nutation can have measured phase differences, taking account of the possible imperfections of the elasticity of the Earth. Nevertheless, any kind of pair arising this way appear only from the gravitational attraction of the Moon on the rigid tide, which is on the order of 10^{-5} of the ascending differences corresponding to the ellipticity of the figure. So similarly, if the "rigid tide" had a lag of 90°, the lag in nutations could be only on the order of 10^{-3} in cyclic measurement. Furthermore, if the lag of the tide is extended even by 1°, this would lead to other conclusions which are not verified. Hence this seems impossible, i.e. that the lag can really be greater than a few seconds of arc in each phase, and it is very unfortunate that it is more than a few. /237

9.5. Conclusions Regarding Comparison of Theory and Observation

If we compare the results of theory and observation, we find that as far as forced polar motion is concerned, the motion has a period ≈1 (1.079) sidereal day, but the radius of the circle has twice the theoretical value. On the other hand, from observations a second cyclic motion results with period 0.932 sidereal day, which would be impossible if the orbit of the pole had the theoretical value of the radius → 0".0006. As far as the values of the constants are concerned, their results differ from the theoretical ones, and the value of N has not yet been exactly determined.

Therefore we must advance our hypotheses. Thus, we consider the relations between the crust and the core of the Earth.

This consideration constitutes the state of the art, as far as we know, on this subject. It must be noted that we do not know positively even if the core is fluid or solid. But we have definitely computed quantities such as density, mass of the core, etc., with satisfactory accuracy. /238

A preview of this theory was given before; in the following, we will develop it in more detail. But in order to complete the comparison between theory and observation, we must say that the conclusions reached up to the present regarding this theory are not enough. Still, a complete theory about the interior of the Earth has not been developed. Some assumptions have simply been put forward on the basis of the results. It is too early to predict whether, even after completion of the theory, some disagreements would exist. In this case, we should search for something more. (At this point, the idea of plasticity is suggested. I.e., if Hook's law of linear dependence is not valid, we will have a remaining permanent deformation. Therefore, the inertia tensor, as has already been developed, will be of the form $I = I_0' + I_\omega + I_1 + I_2$ where I_0' relates to the deformed Earth. This is a subject for future research.)

10. SOME RESULTS CONCERNING THE INTERACTION BETWEEN THE CORE AND THE CRUST OF THE EARTH

10.1. Historical Introduction

The results of the previous chapters show that indeed there exist some disagreements between the results of the theory of rotation of an ideally elastic Earth and the data of the astronomical observations. The reason for these disagreements must be sought in the assumptions incorporated in the theory, and the Earth may, after all, not be considered elastic. The next step is, naturally, to investigate some assumptions about /239
the mechanical properties of the Earth. In order to decide which assumptions we will investigate, we must first consider all the combinations of information that we have for the interior layer of the Earth which can presently be considered trustworthy.

The fact that transverse seismic waves either do not pass through the core of the Earth or, if they pass through, are very attenuated, was interpreted by many geophysicists as a proof that the core is in a fluid state. This, of course, gives rise to the effect of the fluidity of the core on the rotation of the Earth. The first person who seriously strove in this direction was Hopkins in 1830, but his work was published in 1839, 1840 and 1842 and is now of historical value only. Later, W. Thomson considered the problem, and he proved that the result would considerably increase the 1/2-year and chiefly the 14-day nutation. His result was published without proof, and it was verified by subsequent researchers.

The problem of a rotating Earth constituting a rigid shell and a fluid nucleus was first considered with the necessary rigor and completeness by Sloudsky, who made use of the previous work of Joukofsky. At the same time as Sloudsky, Hough considered the same problem, but investigated only free wobble.

In 1910, Poincaré published his research on the precession of a deforming Earth. Considering the case of a rigid shell and a fluid nucleus, he arrived at equations which differed only in form from those of Sloudsky.

From 1910 to 1948, as far as we know, nothing was published /240
on the effect of the fluid nucleus on the rotational motion of the Earth. In the interim, the development of seismology gave basic information about the interior constituents of the Earth. Our conception of the nature of the core changed in an essential way. The hypothesis of a fluid core was initially suggested in order to explain volcanic explosions and the geothermal scale. During the last 100 years, supporters of this hypothesis considered the Earth a fluid mass with a fine crust whose thickness was on the order of 10 km. Now we accept a solid crust of

about 2900 km thickness with a fluid core in it. We have reliable information about the size, the mass, the density, and the compressibility of the core. This made it possible for Jeffreys to give, in two recent works, not only a qualitative, but also a quantitative calculation of the dynamic result of the nucleus. He had in mind the explanation of the known disagreement between the theoretical and observed values of the nutational constant (N). In his first work, he used the equation of Lamp and the values of the moments of inertia of the nucleus and the shell, found by Bullen, and he ignored the elastic deformation of the shell. With these assumptions, he found that the constant of nutation is smaller than the value it would have for a rigid body. While later, from the work of Spencer Jones, it was found to be equal to 9".227, for the theory of an Earth with fluid core, it was found to be equal to 9".172.

In the second work, Jeffreys took account of the effect of the elasticity of the shell and obtained the theoretical value 9".181. Moreover, he found that the coefficients of the main nutation in obliquity and longitude are affected to a different degree by the nucleus, so that the ratio of the axes of the ellipse of nutation needs a correction Δn where Δn = = -0".003. However, the results of the analysis do not assure the later conclusions, even though the correction Δn found by Jeffreys is great enough to be found in the analysis of the observations. /241

This last work of Jeffreys is apparently the only one which attempts to take into account both the motion of the nucleus and the elastic deformation of the Earth. In this last work, Vicente and Jeffreys considered two models for the nucleus, both arranged in such a way as to have the correct mass and moment of inertia and ellipticity given from Bullard's theory about the shape of the Earth.

In the one model, the nucleus is considered homogeneous and incompressible with a separate body in the center. In the other, the nucleus is considered as having a square law for the density, and there is a total change due to pressure. In reality, the known compression would be taken into account for about half the reduction from uniformity of the density. Referring again to the chapter on calculating the main nutational terms, we observe that the results finally obtained agree more with the first model. This is surprising, because the pressure in the nucleus is considered to be great. An intermediate density law which would be more plausible from the geophysical point of view would fit with enough conditions. In the works of Joukowsky and Sloudsky, efforts were made to take into account the viscosity of the nucleus. This problem has recently been considered by Sekiguchi and by Bondi and Lyttleton. The authors of later works used the methods of the matrix theory of motion of a viscous fluid, but some, like Sekiguchi, considered only the case of a spherical nucleus. Note that the case of a viscous fluid is related to the deformation of solid bodies and the moment of fluids. From this brief summary, it is apparent that the study of the effect of the nucleus of the Earth on rotational motion comes down to the solution of particular problems. We do not as yet have a general theory developed to the point where a rigorous test can be made for comparison with the observations. In any case, for the future development of such a theory, it is useful to know that small parts of the particular cases observed have been explained. /242

10.2. Determination of the Moments of Forces Applied on the Crust from the Core

The results allow us to obtain some conclusions about the forces applied on the crust from the core. Without knowing the nature of these forces, we will attempt to define their moments.

The equation of the rotational motion of the crust can be written as

$$\overline{G}_S = \overline{L}_S + \overline{M} \tag{10.2.1}$$

where $\overline{G}_S$ is the angular momentum, $\overline{L}_S$ is the torque of the forces due to the Sun and Moon, $\overline{M}$ the torque of the forces due to the core. We denote by $\overline{G}$ the total angular momentum of the Earth. The derivative of $\overline{G}$ with respect to time is, as we have already seen, a vector on the plane of the equator $(\bar{G} \wedge \dot{\bar{G}} = c)$. Thus we can represent it by a complex quantity: /243

$$\dot{\bar{G}} = G(\sin\vartheta\,\dot{\psi} + i\,\dot{\vartheta}) \tag{10.1.2}$$

We have already met this formula in the development of the modern theory, but there $\dot{\overline{G}}$ was put in the formula of the cross-product $\wedge$ and in front of the first term of the right-hand side, a (-) sign existed. This can be explained by the fact that we used a right-handed system of axes in this chapter. Now we use the more common method, i.e. we take a right-handed positive measurement, so we change the sign of $\dot{\psi}$. We write the following theoretical expressions for the sum of the principal and the 14-day term in longitude and obliquity:

$$\sin\vartheta\psi = -6''.869 \sin \Omega - 0''.0812 \sin 2\mathbb{C}$$
$$\vartheta - \vartheta_0 = 9''.220 \cos \Omega + 0''.0884 \cos 2\mathbb{C} \qquad (10.2.3)$$

For obvious reasons, we use the value of the nutational constant based on the known relation between this constant, the ellipticity H, and the proportionality of the mass of the Moon to the mass of the Earth.

$$N = 231981''.8H \frac{\mu}{1+\mu} \qquad (10.2.4)$$

We take N = 9".220, and with this value we compute the other coefficients in (10.2.9). We obtained them in a greater number of figures than the first, because the observations give the expression for the 14-day term more accurately.

If no interaction existed between the core and the crust, so that torque $\overline{M}$ in (10.1) could be taken as zero, we would have

$$\dot{\bar{G}}_s = h\, G_s\, (\sin\vartheta\dot{\psi} + i\dot{\vartheta}) \qquad (10.2.5)$$

where h is the ratio of the dynamic ellipticity of the crust to that of the Earth as a whole. Then (10.5) can substitute for $\overline{L}_s$ in (10.1). /244

Because the observer is always on the crust, the equation of nutation was obtained from astronomical observations and it describes the vector $\overline{G}_s$ exactly. Here we will consider only the term of period 18 1/2 years and 14 days of this motion. The expression for the main nutational term can be obtained from the coefficients and corrections found previously, or we can find it from the chapter on determination of the main nutational terms, that it is

$$\sin\vartheta\psi_r = -6''.850 \sin \Omega + 0''.008 \cos \Omega + F_\psi$$
$$\theta_r = 9''.198 \cos \Omega + 0''.001 \sin \Omega + F_\theta \qquad (10.6)$$

where F_ψ, F_θ are the sums of all the remaining terms entering the formulas. Then, in order to find the 14-day term, we make use of the results from the chapter on forced polar motion, for the lunar diurnal term (first method of solution). We will have the following equation:

$$\Delta\varphi = 0''.0086 \sin(2\text{☾}-\alpha) - 0''.0019 \cos(2\text{☾}-\alpha) + 0''.0021 \sin(2\text{☾}+\alpha) \tag{10.7}$$

From this we obtain the theoretical expression for the lunar diurnal term which we found in the modern theory for an elastic Earth, i.e.,

$$0''.0051 \sin(2\text{☾}-\alpha).$$

The remainder is:

$$0''.0035 \sin(2\text{☾}-\alpha) - 0''.0019 \cos(2\text{☾}-\alpha) + 0''.0021 \sin(2\text{☾}+\alpha) \tag{10.8}$$

We can now consider the difference between the two expressions for the 14-day nutation of the crust. This is adapted for reduction to the apparent position, and the result is obtained /245 according to astronomical observations. This was found by the following considerations. The equation of motion from the angular momentum of the Earth in space remains the same for any hypothesis concerning the mechanical properties of the Earth, but this is not necessarily true for the angular momentum of the shell alone. Inversely, since the shell is an elastic body, we can take it that vector $\overline{G}_s$ behaves with respect to the shell according to the theory of rotation of an elastic body, i.e. according to the equations x ..., y ... of the modern theory (projections of $\overline{G}$ on $OX_0Y_0Z_0$). Now, when we consider the motion of the shell, we do not need to change the theoretical equations for the forced latitude change and the difference between the theoretical and observational results can be attributed to the inaccuracy of the theoretical expression for the 14-day nutation of the shell, which we write as follows, taking the values of the coefficients given by Woolard:

$$\Delta\delta = -0''.0811 \sin 2\text{☾} \cos\alpha + 0''.0884 \cos 2\text{☾} \sin\alpha \tag{10.9}$$

Taking (10.8), we have from (10.9):

$$(-0''.0866 \sin 2\text{☾} + 0''.0019 \cos 2\text{☾}) + \cos\alpha\,(0,0894 \cos 2\text{☾} + 0,0019 \sin 2\text{☾}) \tag{10.10}$$

By combining this result with (10.6), and by excluding all the remaining nutational terms (since they were excluded in the following transformations), we find:

$$\sin\vartheta\psi_r = -6''850 \sin\Omega + 0''008 \cos\Omega - 0''0866 \sin 2\zeta + 0''0019 \cos 2\zeta$$

$$\vartheta_r = 9,198 \cos\Omega - 0''001 \sin\Omega + 0''0894 \cos 2\zeta + 0''0019 \sin 2\zeta \qquad (10.11)$$

It is worth noting at this point that Fedorov, whose analysis results we take account of in the formulas, found an error in the above expression. However, omitting a slight correction, we will continue as if it were correct.

We can already determine the $\dot{\overline{G}}_s$ by substituting Eqs. (10.11) in the previously given expression /246

$$\dot{\overline{G}}_s = G_s (\sin\vartheta\dot{\psi}_r + i\dot{\vartheta}_r)$$

Because the vector $\overline{M}$ lies on the plane of the equator, it can be written as X + iY, and using (10.1), we have

$$X + iY = G_s [\sin\vartheta(\dot{\psi}_r - h\dot{\psi}) + i(\dot{\vartheta} - h\dot{\vartheta})] \qquad (10.12)$$

It is interesting to compare the above expression to what we will obtain for integrated connections between the nucleus and the shell, where we consider a solid nucleus. The torques of the forces exerted in this case by the nucleus are denoted by X' + iY'. In this case, $\overline{G}_s$ and $\overline{G}$ are practically colinear. Then, by substitution in (10.2), we must obtain the theoretical expression for nutation (10.3). Then we obtain:

$$X' + iY' = G_s (1-h)(\sin\vartheta\dot{\psi} + i\dot{\vartheta}) \qquad (10.13)$$

Then, in order to calculate h, we use the following in the moments of inertia of the nucleus A_n and C_n, obtained by Bullen

$$\varepsilon = \frac{C_n - A_n}{A_n} = 0,0026 \qquad \eta = \frac{A_n}{A} = 0,112 \qquad (10.14)$$

and we find h = 1.027. Substituting (10.3) into (10.13) and writing $\Omega = \alpha t$, $\zeta = \beta t$ where n is the known velocity of the rotation of the Earth, we find:

$$X'+iY' = \alpha G_s\,(0{,}''127\ e^{i\alpha t} - 0{,}''032\ e^{-i\alpha t}) + \beta G_s(0{,}''0023\,e^{i\beta t} - 0{,}''0001\,e^{-i\beta t}) \tag{10.15}$$

The moment of the force caused by the main nutation can be written as the sum of the two vectors

$$\bar{U}_1 = 0{,}''127\,\alpha\, G_s\, e^{i\alpha t}, \quad \bar{U}_2 = -0{,}''032\,\alpha\, G_s\, e^{-i\alpha t}$$

Since α is negative, the first vector rotates on the plane in the right-hand direction, and the other in the left-hand direction. Relative to the Earth, the two vectors rotate with angular velocities $-n + \alpha$, $-n - \alpha$. The torqe caused by the 14-day term can also be expressed as the sum of the vectors /247

$$\bar{V}_1 = 0{,}''0023\,\beta G_s\, e^{i\beta t}, \quad \bar{V}_2 = -0{,}''0001\,\beta G_s\, e^{-i\beta t}$$

In the following, substituting (4.3) and (4.11) in (10.12), we have:

$$\begin{aligned} X+iY &= \alpha G_s\,(0{,}''236\,e^{i\alpha t} - 0{,}''035\,e^{i\alpha t} + 0{,}''004\,e^{i\alpha t} - 0{,}''004\,i\,e^{-i\alpha t}) + \\ &+ \beta G_s\,(-0{,}''0010\,e^{i\beta t} - 0{,}''0024\,e^{i\beta t} + 0{,}''0019\,e^{i\beta t}) = \\ &= (1{,}09+0{,}02i)\,\bar{U}_1 + (1{,}09+0{,}13i)\,\bar{U}_2 + (-0{,}43+0{,}83i)\,\bar{V}_1 + 24\,V_2 \end{aligned} \tag{10.10}$$

We do not think that it is possible, based on these results, to give a quantitative measure of the effects of the fluidity of the nucleus on the interaction between nucleus and shell. However, some quantitative conclusions can be drawn. Among them, the following seem to be the more reliable:

1. The norm of the vector $\bar{u}_1$ increases.

2. The vector $\bar{v}_1$ reverses its direction.

3. Vectors $\bar{u}_1$ and $\bar{u}_2$ diverge in the direction opposite that of rotation of these vectors relative to the Earth.

10.3. Comparison with the Theory of Sloudsky and Poincaré

Results 1 and 2 given in the previous chapter seem at first sight to contradict each other. In any case, this contradiction can be easily explained if we similarly use the simpler formula of the theory of the rotation of the Earth with a fluid nucleus as developed by Sloudsky and Poincaré. This is a reference to the hydrodynamic study of Lamp.

We will use the equations given by Lamp for the calculation of the torqué.

$$A\dot{\bar{\omega}} + F\dot{\bar{\omega}}_n - i(C-A)n\bar{\omega} + iF_n\bar{\omega}_n - ke^{i\sigma t}$$ /248

$$F\dot{\bar{\omega}} + A_n\dot{\bar{\omega}}_n + iC_n n\bar{\omega}_n = 0 \tag{10.17}$$

where $\bar{\omega} = p + iq \quad , \quad \bar{\omega}_n = p_n + iq_n$ (10.18)

where p, q are the components of the angular velocity of the rotation of the Earth around a perpendicular equatorial axis rigidly attached to the shell. p_n and q_n are the components of the angular velocity of the nucleus relative to the shell. Note that here by the term "rotation" we mean "elliptic rotation" (as Joukowsky used the term). F is the magnitude (quantity) of the dimensions of one moment of inertia, and in this case it equals $A_n\sqrt{1-\varepsilon^2}$.

We put $$\nu = \frac{C_n - A_n}{C-A} \tag{10.19}$$

and then (10.17) transforms into

$$A_s\dot{\bar{\omega}} - i(C_s - A_s)n\bar{\omega} = (1-\nu)ke^{i\sigma t} + X + iY.$$

where C_s and A_s are the principal moments of inertia of the shell.

Hence we find

$$\rightarrow X + iY = \nu ke^{i\sigma t} - A\dot{\bar{\omega}} - F\dot{\bar{\omega}}_n + i(C_n - A_n)n\bar{\omega} - iF_n\bar{\omega}_n \tag{10.19}$$

If there is no relative motion of the nucleus → $\bar{\omega}_n = 0$, and

$$X' + iY' = \nu ke^{i\sigma t} - A_n\dot{\bar{\omega}} + i(C_n - A_n)n\bar{\omega} \tag{10.20}$$

The solution of (10.17) is

$$\bar{\omega} = -\frac{A_n\sigma + C_n n}{\Delta(\sigma)}\, i\ell e^{i\sigma t}, \quad \bar{\omega}_n = \frac{F\sigma}{\Delta(\sigma)}\, i k e^{i\sigma t} \qquad (10.21)$$

where

$$\Delta(\sigma) = \begin{vmatrix} A\sigma - (C-A)n & F(\sigma+n) \\ F\sigma & A_n\sigma + C_n n \end{vmatrix}$$

Substituting the values of $\bar{\omega}$ and $\bar{\omega}_n$ in (10.20) and partially rearranging, we have

$$\frac{X}{X'} = \frac{Y}{Y'} = 1 + s\,(\sigma+n), \quad s = \frac{F^2\sigma}{S} \qquad (10.21)$$

$$S = \nu\,\Delta(\sigma) - (A_n\sigma + C_n n)\left[A_n\sigma - (C_n - A_n)n\right]$$

Using (10.14), we can express the moments of inertia of the nucleus in terms of the greater moment of inertia of the Earth, as follows:

$$A_n = \eta\,(1-\mu)\,C = 0{,}1116\,C, \quad C_n = \eta\,(1-\mu)(1+\varepsilon)\,C = 0{,}1119\,C$$ /249

$$F^2 = \eta^2(1-\mu)^2(1-\varepsilon^2)\,C^2 = 0{,}0124\,C^2, \quad \nu = \frac{\eta(1-\mu)}{\mu}\,\varepsilon = 0{,}0884$$

Substituting the above arithmetic values in (10.21) (the first), we obtain for the main and the 14-day nutational term the following results:

Quantity	Main Nutation (Vector $\bar{u}_1$)	14-Day Nutational ($\bar{v}_1$)
σ	$(-1 - 0{,}000146)\,n$	$(-1 + 0{,}073)\,n$
$\Delta(\sigma)$	$-0{,}00028\,\eta^2 C^2$	$0{,}00695\,\eta^2 C^2$
S	$0{,}00001\,\eta^2 C^2$	$0{,}00026\,\eta^2 C^2$
sn	-1400	-44
$1 + s\,(\sigma+n)$	$1{,}20$	$-2{,}21$

We therefore observe that the result of the fluidity of the nucleus is an increase in vector $\bar{u}_1$ and an inversion of direction of $\bar{v}_1$. It is natural to assume that the divergence of $\bar{u}_1$ and $\bar{u}_2$ in the direction of the diurnal rotation of the Earth can be explained by friction at the boundary of the nucleus.

Considering a summary of the above, we note that the final assumption we make is a solid crust of 2900 km thickness and a fluid core. The most complete work in this direction was done by Jeffrey in two successive papers. In the second, he considers the motion of the nucleus as well as the elastic deformation of the shell. That is, the theory of a totally elastic Earth is not valid any longer. However, no complete theory of a fluid nucleus was developed, but we had only a few points of disagreement. In the following development, an attempt is made to determine the moments of the forces which act on the shell from the nucleus. The investigation is based on the relation /250 $\dot{\bar{G}}_s = \bar{L}_s + \bar{M}$ where $\bar{G}_s$ is the angular momentum, $\bar{L}_s$ is the moment of the forces due to the Sun and Moon, and M is the moment of the forces due to the nucleus.

We propose the following:

If no interaction existed between shell and nucleus, then it would be

$$\dot{\bar{G}}_s = h G_s (\sin\vartheta \dot{\psi} + i \dot{\vartheta}) \quad \text{where } \bar{G} = G(\sin\vartheta \dot{\psi} + i\, n\mu\, \vartheta)$$

and h = ratio of the dynamic ellipticity of the shell to that of the entire Earth, i.e. we define a corresponding expression of $\bar{G}$.

But in this case, it will also be

$$\bar{M} = 0 \longrightarrow \bar{L}_s = h\, G (\cos\vartheta \dot{\psi} + i \dot{\vartheta})$$

Note that the observer stands on the shell. Thus the observations essentially describe $\bar{G}_s$ and not $\bar{G}$, as was considered up to now.

We already know $\bar{L}_s$. If we determine $\dot{\bar{G}}_s$ also, we can calculate $\bar{M}$. But, based on the above remark, $\dot{\bar{G}}_s$ can be taken from the observations. From the chapter on the principal nutational terms, we obtain corresponding expressions for the principal and the 14-day term. Note that as was stated in the modern theory, $\bar{G}_s$ behaves (i.e. it is determined) toward the shell, according to the theory of a perfectly elastic body. Consider M as X + iY (it lies on the plane of the Equator)

$$\rightarrow X + iY = \bar{G}_s - \bar{L}_s = G_s\,[\sin\vartheta\,(\dot{\psi}_r - h\dot{\psi}) + i\,(\dot{\theta} - h\dot{\theta})]$$

Similarly we find the expression for a solid nucleus X' + iY' which up to now has not given the expected results, at least in its present form. /251

If we compare it with Poincaré's theory (that is without final substitution from observations), it again results in an increase of the principal nutation and inversion of the vector of the 14-day nutation. The existing differences can not be attributed to the friction of the nucleus on the shell.

We can not, of course, expect quantitative agreement between the theoretical results and those obtained from observations, because the model of the Earth that we used in the theory is a rough simplification. The elasticity of the shell and the viscosity of the nucleus were not considered in this model (these were later considered by Jeffreys). At the same time, the effect of the nucleus on the shell's motion is probably not defined by the friction on the shell. Also, other kinds of forces, e.g. of a magnetic nature, can play an important role. Note that this is the state of the art on this subject. We see therefore that each theory is simply a better approximation toward the convergence of theory and observation, and also that much work is still required, if not an inspired idea.

11. STUDY OF THE POLAR MOTIONS

11.1. Motions

From the study of the observations according to the methods of elaboration presented in the previous chapters, it was derived that the pole of the Earth does not have a fixed position, but it describes a complicated curve continuously moving on the surface of our planet. This curve is called polar orbit. By an appropriate method, we can analyze this motion, breaking it into other, simpler ones, and therefore consider it as the vector sum of the following separate motions. /252

1. We have two motions of the pole of a periodic nature. The first has a period of 1 year and is thus called annual motion. Due to this motion, the pole of the Earth describes on the surface of the Earth an orbit of more or less elliptic shape, the radius of which changes from year to year between the values 0".06 and 0".10.

2. The second periodic motion has no constant period but exhibits changes as time passes ranging from 412 to 442 days, i.e. it has a period on the order of 14 months. This motion is known as Chandler motion. Because of this, the pole of the Earth describes an orbit of circular shape the radius of which changes with time, taking values between 0".07 and 0".25.

3. Besides the above periodic changes of the instantaneous pole, we have an eternal change of the position of the pole because of which the average position of the pole is displaced on the surface of the Earth by approximately 0".003 per year in the direction of the meridian corresponding to the value of longitude $\lambda = 60°$ W. This result needs further verification.

4. Finally, we have also other changes in the position of the instantaneous pole which probably correspond to periodic motions of small amplitude, or they are totally irregular.

5. The above were found based on the hypothesis that the Earth is not subject to the action of external forces. But in fact, the Earth is subject to the action of the forces of the bodies of our planetary system and especially of the Sun and Moon.

Therefore, due to the action of gravitational forces of these two bodies on the Earth as a whole, we have another motion of the instantaneous pole of the Earth which is superimposed on the previously studied ones. Because of this motion, called lunar-solar motion, the instantaneous pole describes within a day an almost circular orbit, the radius of which /253 changes between the values 0".00 and 0".02. For an observer standing over the North Pole of the Earth, the lunar-solar motion of the instantaneous North Pole of the Earth seems to take place in the retrograde direction, while the annual polar motion and the Chandler motion, and also the final sum of motions seem to take place in the right direction, i.e. in the same direction as the rotation of the Earth. (The matter of the diurnal lunar-solar motion has already been studied.) We will attempt a brief qualitative and quantitative interpretation of the above motions. We must immediately say that the quantitative interpretation is not yet complete. As far as the qualitative one, in general, it is at least satisfactory.

Therefore, the annual motion is most probably caused by the displacements of masses at each epoch in the interior, the oceans and the atmosphere of Earth. I.e., the annual motion is a forced motion due to the change in physical elements. On the contrary, the polar motion with the Chandler period is not forced, but free, corresponding to the condition predicted by Euler, with the difference that its period is greater than the predicted one, i.e. instead of 304 days it is on the order of 14 months. This difference in the value of the period comes from the different consideration of the Earth, i.e. Euler considered the Earth as a perfectly rigid body while the Earth is subject to elastic and plastic deformations. Indeed, it can be proved that if the Earth is a deformable body and it /254
rotates initially around an axis not coinciding with its principal axis of inertia, then it deforms with the resultant increase in the value of the period calculated by Euler.

As far as the eternal motion is concerned, we do not have a clear answer. The problem appears to be extremely difficult, because first of all we must eliminate all the systematic errors which enter in the determination of the position of the instantaneous pole. Another problem which also appears is the distinction between genuine polar motion and the effect of the change in relative positions of the international observatories, due to the phenomenon of displacement of the continents. In any case, for the time being the only thing we can say with assurance is that the total displacement of the average pole does not exceed 1". Note that an eternal change of the average pole on the order of 0".003 per year can be easily interpreted as the result of the class of poles which is observed in the area of Greenland.

Finally, the irregular changes in position of the instantaneous pole are possibly caused by irregular displacements of masses in the interior or on the surface of the Earth as, for example, displacement due to earthquake, volcanic explosions, or due to singular meteorological phenoemna.

Summarizing the above, we have:

1. Free periodic motion of Chandler

2. Annual periodic motion (forced)

3. Lunar-solar diurnal motion

4. Eternal motion

5. Small period motions
   Totally irregular motions

Among the above motions, f r e e periodic motion was investigated fully. Also the annual periodic motion and lunar-solar diurnal motion were statistically developed from observations. In the following, we will investigate eternal motion. /255

11.2. Eternal Polar Motion

We already mentioned that the Earth's pole has an eternal motion. This phenomenon must be differentiated from that of the displacment of continents. The hypothesis that the continents are moving arose from the research of geophysicists during the past years. Enough theories appeared, like the one by Wegener, which assumed that the Eastern and Western hemispheres move independently of each other.

Reliable information about continental displacement is obtained from paleomagnetic observations of rocks, where we examine the direction of the magnetic field of the Earth during the period when the rock was red-hot. Modern geophysical theories seek to explain exactly these paleomagnetic theories. Thus it is shown that two things might have happened in the past: 1. Displacement of the pole on a great scale; 2. continental displacement.

Research was done by some investigators to prove that continental displacement was derived from changes observed in latitude and longitude, and that it is not the same phenomenon. In any case, these changes are possibly caused by observation errors, lack of homogeneity, and by changes which occurred in the total observation. In our time, it is generally accepted that a continental displacement exists, but it has very small value, so that it requires careful calculation of the sytematic errors, mainly those caused by the proper motions of the stars.

Measurements of continental displacement give a mean value for a long time interval. Also, a paleomagnetic check gives about the same value of the displacement velocity. This is on the order of 0.5-0.3 cm/year. But this implies a displacement of 1 meter in about 33 years, with the exception of certain areas in India, which are displaced faster. The probable error of observation in the I.L.S. stations for a year is on the order of 0".022. Therefore, from the observations made by these stations for 1 year, we can not discover this phenomenon. For this purpose, observations over a long sequence of years, on the order of 50 years, are required in order to have significant results. /256

Since we are concerned with continental displacement, it is obvious that we will obtain better results from observations of longitude. Because of this idea, systematic observations were made using the astrolabe and the PZT of longitude. The general conclusion of the above work is that the change in longitude is not caused only by continental displacement. After that, the phenomenon of eternal polar motion will be considered separately from continental displacement. For the study of this phenomenon, analysis of the observations of the I.L.S. stations for the last 66 years was done. The analysis showed that the mean pole has an eternal motion which is composed of a "progressive" component on the order of 0".0035/year (10 m along the 65° W meridian) and of an "equilibrated" component (oscillating) with a period of 24 years along the 122° W meridian. Note that especially the observations for the eternal polar motion were made by observing the same stars from a chain of stations, which eliminates the errors due to the positions of the stars.

Of the two components above, the one of 24-hour period is rather surprising, because as far as we know there is no geophysical phenomenon of such period. Finally, Anna Stuko observed an increase of latitude of Ukiah by 0".003 and a decrease of latitude of Mizusawa by 0".003 (as much as the "progressive" component), and she considered this as an assertion of the geophysical theory about the left-handed rotation of the shore of the Pacific Ocean. /257

12. LOVE NUMBERS

12.1. Introduction

In the chapter about deformations we stated that investigating problems concerned with oscillations of a year or smaller period, for a plastic Earth, the use of Love's numbers is suggested; these are dimensionless numbers, essentially coefficients, describing different physical phenomena. With the exception of the free periodic motion of Chandler, all the others have periods on the order of 1 year or less. Therefore, it is considered suitable to develop a method of investigation by Love's numbers. To this end, we transform appropriately the Euler equations into a form given by Liouville. Then we develop the theory of Love numbers after making reference for a time to the reference figures. By the Love numbers, we transform the initial equations of Liouville. Finally, we obtain the form of Liouville's equation for each case, as, for example, for forced oscillation, free oscillation, etc.

Closing the development of this method, we investigate the equation of excitation which moves the pole to different positions around an assumed initial position, and we give a geometric presentation of them. Thus it only remains to apply this method to the eternal motion as well as to motions /258
of small periods. Note that the method of investigation with Love numbers is, so to speak, empirical and more accurate than the others, because the Love numbers exactly represent the real situation of the Earth. The great advantage of this method is that our equations can be adjusted to any future improvement of our knowledge about the elastic behavior of the Earth.

12.2. Liouville Equations

In the chapter on the classical Euler theory, the formula in which the change of angular momentum due to externally applied torque is described is given as:

$$\bar{L} = \frac{dG}{dt}\bar{k} + \bar{G} \wedge \bar{\omega}$$

From this equation, by dt transformations we have the Euler equations

$$L_1 = \frac{dG_1}{dt} - (J_2 - J_3)\omega_2\omega_3$$

$$L_2 = \frac{dG_2}{dt} - (J_3 - J_1)\omega_3\omega_1$$

$$L_3 = \frac{dG_3}{dt} - (J_1 - J_2)\,\omega_1\omega_2$$

If these equations are written in one form, we have the formula

$$\boxed{L_i = \frac{dG_i}{dt} + \varepsilon_{ijk}\,\omega_j\,G_k} \qquad (11.1)$$

where, according to the usual summation convention, each index different from i takes all possible values, and we add the results: if $i \neq j \neq k$, then $(i, j, k) = (1, 2, 3)$.

In the above equation, G stands for the angular momentum, L_i for the externally applied torque, ω for the components of the angular velocity and ε_{ijk} is the corresponding Kröneker symbol for three indices. That is, it will be

$\varepsilon_{ijk} = 0$ if any two indices are equal, $i = j$, $j = k$, $k = i$ /259

$\varepsilon_{ijk} = 1$ if the indices are of the order 1, 2, 3, 1, 2 (even)

$\varepsilon_{ijk} = -1$ if the indices are in odd order 1, 3, 2, 1, 3.

The equations of form (11.1) are very general, for example, they may refer to a system of moving molecules. It is useful for the following to separate the angular momentum into two parts:

$$\vec{G}_i = G_{ij}(t)\,\omega_j + g_i\,(t)$$

where $C_{ij} = \int_V \rho\,(x_k x_k\,\delta_{ij} - x_i x_j)\,dV$

is the variable inertia tensor for the material included in a volume V and δ_{ij} is the Kröneker delta, where

$\delta_{ij} = 0$ if $i = j$

$\delta_{ij} = 1$ if $i \neq j$

$g_i(t)$ represents a relative angular momentum

$$g_i = \int_V \rho \, \varepsilon_{ijk} \, x_j \, u_k \, dV$$

due to the motion u_i relative to the x_i system (rotating). By substitution, we obtain the equation

$$L_j = \frac{d}{dt} \left(C_{ij} \omega_j + g_i \right) + \varepsilon_{ijk} \, \omega_j \left(C_{kl} \omega_l + g_k \right) \tag{11.2}$$

This equation was obtained by Liouville in 1858, and it is called the Liouville equation. All investigations dealing with irregularities in rotation will be obtained as particular solutions of the above equation. We make the following remarks:

1. L_j represents an external torque which acts on a body occupying volume V. The surface bounding this body can be chosen appropriately, for example considering wind, we could exclude the atmosphere, choosing the surface so that in this case an external torque would exist due to the air pressure, or /260
we could consider the planet as a whole, and then it would be $L_j = 0$. The choice is dictated by the total number of available instruments, e.g., it is easier to determine the air pressure than the angular momentum.

On the other hand, L_j is the component of the moments of the forces along the axis of rotation. The change in L_j must be examined with special attention. If, for example, we consider the moment due to wind moving along an axis constant with respect to time, $L_j = 0$. But a torque constant in space has components which vary with time and with diurnal frequency.

2. The differentiation which takes place has been referred to a time 0 when the rotating system x_i and the fixed X_i coincide. At this instant, the components L_i, g_i and C_{ij} in the two systems are the same. But for every future time t' when x_i has changed position, we chose a "fixed" system X_i' the axes of which coincide with those of x_i. These are applied according to the classical theory. In particular, as far as quantities dC_{ij}/dt and dg_i/dt are concerned, it is easier and safer first to make the integration from the formulas of their definitions and then to substitute them in the general formula and carry out the differentiation. This is of great significance when the surface S, which bounds the volume with respect to which we differentiate, is changing, so that if the integration takes place before, only then can we overcome deformation of S.

3. Quantities g_i and C_{ij} depend on the density field $\rho(x_K,t)$ and the relative velocity $u_i(x_K,t)$. In our equations, ρ and u_i appear as independent variables.

There are certain restrictions imposed by the conservation of mass, energy, and momentum. However, the equations could be transformed so that they are not constant but are satisfied with respect to conservation, e.g. change in the density field $\rho(t)$ for motion $u_i(t)$. In the following exposition, the fields ρ and u_i are defined for each geophysical application, according to their particular laws. With L_i, g_i and C_{ij} completely defined, the equations can be solved for angular momentum $\omega_i(t)$ of the reference system x_i relative to the fixed system X_i. /261

Consider a coordinate system Oy_i rotating with angular velocity $\Omega = (\omega,\omega_i)^{1/2}$ as the x_i system but with axis y_3 directed along the instantaneous axis of rotation. Then

$$x_i = \frac{\omega_i}{\Omega} y_3$$

ω_i/Ω represents the direction cosines of the rotating axes with respect to the reference axes. (Note that if d/dt represents the acceleration of diurnal rotation, then ω_1, ω_2 are the components of the wobble.)

12.3. Reference Coordinate Systems

Some distinction is made by some authors between the rotating axes of reference in the Euler equations and the body axes of the Earth, the changes of which can be described by:

$$C_{ij} = \int_V \rho\,(x_k x_k \delta_{ij} - x_i x_j)\,dV\,, \qquad g_i = \int_V \rho\,\varepsilon_{ijk}\,x_j\,\dot{x}_k\,dV$$

The two rotating systems can be combined with no loss of generality. The choice of the x_i system is totally arbitrary, e.g. it could rotate with an angular velocity in the opposite direction from the direciton of the rotation of the Earth. /262
However, for a coordinate system to be more suitable, the coordinate axes must be attached somehow to the Earth. In most papers dealing with the coordinate systems, the rotation takes place "simultaneously with the rotation of the Earth."

If the Earth were perfectly rigid, there would be no further difficulties. But wind, ocean currents, and the fluid core cause complications. For that reason, the axes can be fixed to the "rigid Earth." But there are still tidal deformations of the solid crust. On the other hand, a relative motion of the shell of the Earth is accepted by geology. Such a motion is known to take place, and it has been accepted by Wegener as a displacement of one continent relative to another. Thus, finally, we require a set of fixed axes which are kinematically defined so as to have no restriction on the deforming Earth. There is a number of possible choices.

1. The Mean Axes of the Body (of the Earth) of Tisserand

These are defined in such a way as to have $g_i = 0$. Thus, if the wind, ocean currents, and all other relative motions stop, these axes will rotate with the resultant rigid body. For a perfectly rigid body, rotating with angular velocity ω_i, the velocity of any material point is the vector

$$dx_i/dt = \varepsilon_{ijk}\, \omega_j x_k$$

For a deforming body, we can choose a value of ω_i, e.g. $\bar{\omega}_i$, which minimizes the quantity

$$\int_V \rho \left(\frac{dx_i}{dt} - \varepsilon_{ijk}\, \omega_j x_k\right)^2 dV$$

This can be proved for the $\bar{\omega}_i$ axes so that $g_i = 0$, where $\bar{\omega}_i$ is the angular velocity of the mean axes. Jeffreys, whose work we already mentioned, refers the calculations to the mean axes. /263

2. The Principal Axes or the Axes of the Figure

These are defined in such a way as to have zero product of inertia C_{ij} $i \neq j$. Darwin chose the principal axes for his study of the pole.

The differentiation in the previous chapter is referred to any set of perfectly rigid axes of rotation. Therefore, both the mean and the principal axes are included in special cases.

Relations $g_i = 0$, $G_{ij} = 0$, $i \neq j$ are the obvious choices for mathematical simplification, and they lead to noteworthy simplification of Liouville's relations. But there are disadvantages in these basic axes. The wind and other relative motions rotate the mean axes slowly relative to the observatories, and this must necessarily be taken into account in the correction of the observed values. Similarly, the principal axes move relative to the observatories. Therefore, Jeffreys' choice does not facilitate consideration of the facts of relative motion, so he ignores them. Also, change in angular momentum of the atmosphere displaces the mean axes according to the equinoctial swelling of the Earth, and therefore we will have smooth changes of the tensor C_{ij}, if the Earth is perfectly rigid.

3. The Geographical Axes

For all the above reasons, the use of the geographical axes was found necessary; these are fixed in a prescribed manner to the observatories. There are difficulties due to the relative motion of the observatories. For many problems, the relative motion can be neglected. The I.L.S. stations have been displaced close together. /264

If relative motion is not negligible, we choose a set of fixed axes which we attach in some prescribed manner to the observatories. The geophysical observations, astronomical observations, the relative motion of the observatories, and the equations previously given, etc. are referred to these axes.

The origin of all three systems is placed in the center of the Earth, so that

$$\int \rho x_i \, dV = 0$$

12.4. Love Numbers and Relative Coefficients

If the Earth were perfectly rigid, we could apply the Liouvill equations in order to calculate changes in rotation which arise from special geophysical phenomena. For a deforming Earth, the equations are also applied, but their application is allowed only for secondary phenomena, as, for example, the deformation (displacement) of the Earth under the action of small loads and the displacement of the equinoctial swelling resulting from the changes in rotation. Such displacements of mass must be taken into account at the same time as special displacements. A comparison of the geophysical and astronomical observations would provide us with information concerning the elastic or inelastic properties of the Earth.

The deformations result from "massive" forces such as tidal forces, and from surface forces, such as atmospheric pressure. A load suddenly applied on the surface of the Earth causes elastic waves traveling with a velocity on the order of kilometers per second. Fundamental modes of free vibration of the Earth related to these waves have periods on the order of 1 hour. If the period of the forced function is large compared to this, then it can be assumed that elastic deformations take place instantaneously, and they are given from static considerations. The oceans and the fluid core need greater response time than the free vibrations, and there are certain conditions under which static theory is applied. Consider the correspondence of the Earth to a variable potential U(r)β' in degree from tidal forces due to the Moon and Sun and from centrifugal forces resulting from rotation. These can be written as the gradients of such a potential. The result of the deformation defines the Love numbers as follows: /265

The ground rises by $hU_{surface}/g$ and the portion added from the gravitational potential to the displaced surface arises only after a new distribution, and it is KU. Therefore, (1 + K) is a factor which gives the attraction of the swelling by itself, and the substitution by hU/g takes account of this self-attraction. A fluid surface covering the sphere will remain of equal potential, and it will rise by (1 + K)U/g relative to the center of the Earth and by (1 + K - h)U/g relative to the bottom of the sea.

In addition to the vertical displacement of the solid surface by hU/g, there is a horizontal displacement with components

$$1/g\ \partial U/\partial\theta\ ,\quad 1/g\ 1/\sin\theta\ \partial U/\partial\lambda$$

where θ = 90 - ϕ and λ = eastern longitude.

The Love numbers are dimensionless parameters with which we specify some of the elastic properties of the Earth. Their estimation is the subject of elasticity. Information is taken from a /266
great variety of sequences. The great advantage in writing the equations in terms of these parameters is that the equations can be adjusted to any future improvement of our knowledge concerning the elastic behavior of the Earth. The most detailed calculation is the one by Takechi (1950) based on variations of density and elastic properties of the Earth as they are derived from seismic and other methods. His results are:

$$k_T = 0{,}290 \qquad h_T = 0{,}587 \qquad l_T = 0{,}068$$

$$k_T = 0{,}281 \qquad h_T = 0{,}610 \qquad l_T = 0{,}082$$

for the two models suggested by Bullen. There are other methods of measuring the Love numbers, and for various reasons the results are not immediately comparable. We will distinguish the following cases.

1. Deformation from Rotation

Consider the distortion of the Earth due to angular potential U of degree β'. The distortion constitutes the source of an external gravitational potential $K(\alpha^5/r^5)U$, by the definition of K. But the gravitational potential close to the mantle which insignificantly differs from a body of spherical symmetry is given by the formula of MacCullagh. In the present case, the deformation is a spherical harmonic of β' degree, and the relative term of MacCullagh's formula can be written $(Gm/r) + U$ where

$$V = \frac{G}{2r^5}\left[C_{11}\left(x_2^2 + x_3^2 - 2x_1^2\right) + \dots - 6C_{12}x_1x_2 - \dots\right] = K\frac{\alpha^5}{r^5}U$$

The dots imply two additional terms which are obtained by cyclically interchanging the indices. Consider the special case of centrifugal potential which is equal to $1/2\omega^2$ (the square of the distance from the axis of rotation) or

/267

$$1/2\left[\omega^2 r^2 - (\omega_i x_i)^2\right], \quad \omega^2 = \omega_i\omega_i \qquad r^2 = x_i x_i$$

This can be incorporated in the terms $1/3\omega^2 r^2 + v$ where

$$v = 1/6\left[\omega_1^2\left(x_2^2 + x_3^2 - 2x_1^2\right) + \dots - 6\omega_1\omega_2 x_1 x_2 - \dots\right]$$

is a spherical harmonic of β' degree. The term $1/3\omega^2 r^2$ leads to a purely radial deformation which consists of a contraction near the center of the Earth and an extension at the exterior parts. By substitution of the value of U in V

$$C_{ij} = I\delta_{ij} + \frac{K\alpha^5}{3G}\omega_i\omega_j + \text{constant where } I = 1/3\left(C_{11} + C_{22} + C_{33}\right)$$

is the inertia of the sphere in the absence of rotational deformation. This forms the constant, so

$$C_{ij} = I\delta_{ij} + \frac{K\alpha^5}{3G}\left(\omega_i\omega_j - 1/3\,\omega^2\delta_{ij}\right)$$

G = gravitational constant = $6.670\cdot10^{-8}$

a = radius of the Earth $cm^3g^{-1}cm^{-2}$

r = distance from the center of the Earth.

2. The Secular (Eternal) Love Numbers

The Love number K can be interpreted as a measurement of the displacement of the Earth due to centrifugal deformation in the corresponding sequence during the last 5 billion years. With no loss of generality, we can put the x_3 axis along the vector of rotation. Then

$$\omega_1 = 0,\quad \omega_2 = 0,\quad \omega_3 = \Omega \quad \text{(mean diurnal rotation) and } C_{11} = C_{22} =$$

$$= A = I - \frac{K_1 a^5}{9G}\Omega^2,\quad C_{33} = C = I + \frac{2K_3 a^5}{9G}\Omega^2$$

so $K_3 = 3GHC/\alpha^5\Omega^5$ where H is the precessional constant. If all the mass were concentrated at the center, $C = 0 \rightarrow K_s = 0$. For a homogeneous sphere, $C = {}^2/_5 M\alpha^2$ and with $M - 5.98\cdot10^{27}$ g for the mass of the Earth, we obtain $K_s = 1.14$. The exact value lies between these limits. From the value of H and the universal gravitational law, we obtain:

$$C = 0.3336\ H\alpha^2 \text{ (compare: homogeneous Earth} \rightarrow C = 0.4\ M\alpha^2)$$ /268

So $K_s = 0.96$.

3. The Love Numbers for Fluid [Earth]

The previous calculation of K entails the observed value of precession and the form of the ellipsoid derived from measurement of gravity. There are no considerations concerning the relations force - deformation outside the Earth. Now we will calculate the "fluid" Love numbers based on the hypothesis that the Earth is in hydrostatic equilibrium, i.e. that it has the shape of a rotating fluid with the same density as that of the real Earth. For a first order approximation, the ellipticity of the surface is given by

$$\varepsilon = h_f\left(\tfrac{1}{2}\Omega^2 \frac{a}{g}\right)$$

If the entire mass were concentrated at the center, it would be $h_f = 1$ and $\varepsilon = 1/580$. For a homogeneous Earth, Kelvin showed that $h_f = 5/2$, so that $\varepsilon = 1/232$. The observed ellipticity

1/297 lies between these values. By using the observed value of ε, we have $h_f = 580/297 = 1.96$.

But for a fluid surface $h_f = 1 + K_f \rightarrow K_f = 0.96$, which is arithmetically equal to the eternal Love number K_s. A more precise determination was given by Bullard (1948). From the observed precession and the distribution of the density found by Bullen in the Earth, Bullard obtained $\varepsilon^{-1} = 297; 338 \pm 0.050$, assuming hydrostatic equilibrium. The eccentricity can be independently derived, without the assumption of hydrostatic equilibrium, from the observations of gravity or from the motions of the Moon. The resulting values, 296.17 ± 0.68 and $296.72 \pm$ ± 0.65 do not differ significantly from the previous value. The more recent analysis of the observations of gravity (Heiskanen, 1958) gave 297.0 to 297.2, and they are in better agreement with /269
the hydrostatic value. According to these observations, the shape of the rotating Earth is not different from that of the equivalent rotating fluid. It could be a difference of the order of about 1-3%, and in f a c t observations from satellites showed that this occurs. Such a difference, if it were real, would be a measurement of how long nonmatieral forces are applied on the Earth which can withstand deformation under such forces due to its final rigidity. In the case of infinite rigidity $\rightarrow K_f \neq K_s$. The question whether $K_f \approx K_s$ or $K_f = K_s$ is important.

4. The Love Numbers Resulting from the Tide

From studies of the Earth's tides and of the wobble described by Chandler, $h = 0.59$ and $K = 0.29$ are obtained. The close agreement with the values previously given by Takenchi obtained from seismic observations is exceptional. A difference is noticed in the values $h_s = 1.96$, $K_s = 0.96$, obtained from the shape of the Earth. A number of hypotheses can explain this difference, and it is not known which of them is right. One hypothesis is based on the relative magnitude of the pressure to the eternal Love numbers which are referred to the pressure differences, above a limiting force, or to the Love numbers due to the tide, which are referred to the pressure differences under a limiting force.

A second hypothesis is based on the relative duration of the pressures, and a third on the assumption that the Earth was initially in a melting state and that now it has the shape of the bodies at the time of melting. Then, the agreement between K_s and K_f entails a small change or no change in rotation. This is very disagreeable. But there is no doubt that the Earth cor- /270
responds in a different way to the usual tidal potential and the annual wobble corresponds to the diurnal rotation.

We can consider h, K as the asymptotic cases of the generalized Love numbers when the frequency is high and the perturbation infinitesimal. Also, h_s and K_s are considered the asymptotic values for small frequency values and large amplitudes. The Love numbers due to the effects of the tide are taken from a total correspondence of the planet Earth to a perturbation potential. This is given for the combination of the nucleus, the shell, and the displacement of the oceans. The change in inertia due to deformation caused by rotation is obtained from the formula

$$C_{ij} = I\delta_{ij} + \frac{K a^5}{3G}\left(\omega_i \omega_j - \frac{1}{3}\omega\delta_{ij}\right)$$

The important terms are the products of inertia, which are obtained as:

$$C_{ij} = \frac{K}{K_f}\,\frac{\omega_i \omega_j}{\Omega^2}\,(C-A) \qquad i \neq j$$

$i \neq j$, putting $K_s = K_f$.

5. The "Equivalent" Earth

Kelvin showed that for an incompressible homogeneous sphere of stiffness $\bar{\mu}$, the following is true:

$$h = \frac{5/2}{1+\mu}\,, \quad K = \frac{3/2}{1+\mu}\,, \quad \ell = \frac{3/4}{1+\mu} \qquad \mu = \frac{19}{2}\,\frac{\bar{\mu}}{\rho g a}$$

Kelvin's relation gives the values $K_f = 2.5$, $K_f = 1.5$ for the Love numbers, as compared to the observed values $h_f = 1.96$, $K_f = 0.96$ for the Earth. (There are no observed values for ℓ_f.) For a better adjustment to the real conditions, the simplest way (method) is to take the relations $h = h_f/(1+M)$, $K = K_f/(1+M)$, which have the same formula as the Kelvin solution, but they use the observed values $h_f = 1.96$, $K_f = 0.96$. The usefulness of the model of an "equivalent" Earth depends on how well the two known values h and K can be calculated by a suitable choice /271
of zenith parameter μ.

The set of values $\mu = 2.3$, $h = 0.59$, $K = 0.29$ is in exceptional agreement with the best calculations of the stiffness due to the tide. This value, $\mu = 2.3$, is a satisfactory measurement of the tidal stiffness of the Earth. In conclusion, we can roughly compute the "fluid" Love numbers ℓ_f to be 0.23 as compared to 3/4 for a homogeneous Earth. For this value, and for $\mu = 2.3$, we obtain:

$$\ell = \frac{\ell f}{1+\mu} = 0.07$$

in agreement with Takechi's calculations.

6. The Love Numbers of n Degree

We will have the opportunity to find the Love numbers in every degree n. Their definitions are given simply by writing U_n in the first chapter instead of U. For a homogeneous asymptotic sphere, the formulas are

$$h_n = \frac{n+1/2}{n-1}\,\frac{1}{1+\mu N}\,,\quad k_n = \frac{3/2}{n-1}\,\frac{1}{1+\mu N}\,,\quad \ell_n = \frac{3/2}{n(n-1)}\,\frac{1}{1+\mu N}\quad \text{where } N = \frac{2(2n^2+4n+3)}{19n}$$

for n = 2 → N = 1, and we will have the formulas of the equivalent Earth.

7. Fluid Earth and Plane Stress (or Tension)

Another interesting case is that of a thin elastic crust on a liquid Earth. Perhaps this case is well covered by considering a liquid sphere with a plane strain $\bar{u}$ which arises from the curvature. Logically the crust is not under strain unless it is distorted (deformed). Consider u_2 any potential of β' degree due to perturbation. The deformed surface is at $r = \alpha(1+\varepsilon S_2)$ where

$$S_n = P_n^m \cos\vartheta\ (\cos m\lambda, \sin m\lambda) \quad m = 0 \text{ to } n$$

is the surface harmonic and p_n^m is the relative Legendre function defined by the relation /272

$$P_n^m \cos\vartheta = \frac{(n-m)!\,(n+m)!}{2^n}\cdot \sin^m\vartheta \sum_{r=0}^{n-m} \frac{(\cos\vartheta-1)^{n-m-r}(\cos\vartheta+1)^r}{(m+r)!\,(n-r)!\,(n-m-r)!\,r!}$$

The potentials resulting from the superposition of the plane strain and the deformation of the sphere are:

$$V = -\frac{3\bar{u}}{\rho r^2}\,\alpha\varepsilon S_2\,,\quad W = \frac{4}{3}\pi G\alpha^2\left(\frac{1}{r} + \frac{3\alpha^2\varepsilon S_2}{5r^3}\right)$$

p = density, r = distance from the center of the Earth, G = constant of gravity, a = radius of the Earth, ε = ellipticity, S_2 = spherical surface harmonic of degree 2.

For a homogeneous sphere 3g - 4πGαp, $u_2 + v_2 + w_2$ = constant on the plane strain and the terms including S_2 give

$$h = \frac{(U_2/g)}{a\varepsilon S_2}\text{surf.} = -\frac{(V_2+W_2)}{g\varepsilon S_2}\text{surf.} = \frac{5/2}{1+\upsilon}, \quad K = \frac{3/2}{1+\upsilon} \quad \text{where} \quad \upsilon = \frac{15}{2}\frac{\bar{\upsilon}}{\rho g a^2}$$

which is a dimensionless surface tension entering in the same way as the dimensionless stiffness $\bar{\mu}$.

8. Love Operators and Love Complex Numbers

In the study of polar motion and the attenuation of the Chandler wobble, we will make use of the solutions in which the Earth is considered as a Maxwell or Kelvin-Voigt model. For a Maxwell, or viscoelastic (elastoplastic) model, the total value of the deformation is written as the sum of an elastic and a viscous term:

$$\rightarrow \frac{1}{S}\frac{dS}{dt} = \frac{1}{2\bar{\mu}}\frac{d}{dt}(\tau_{\text{elastic}}) + \frac{1}{2\bar{\eta}\mu}(\tau_{\text{elastic}})$$

where μ is the stiffness, $\bar{n}$ the dynamic viscosity and $\tau_{elastic}$ the elastic stress. For a Kelvin-Voigt model, the total stress is written as a sum of an elastic and a viscous stress

$$\tau_{\text{elastic}} = 2\bar{\mu}\varepsilon + 2\bar{\eta}_{K-V}\frac{d\varepsilon}{dt}$$

The Kelvin-Voigt model is characterized by the fact that there is no constant stress associated with the deformation. The Kelvin-Voigt model is represented as a jump and oscillation damping in parallel, while the Maxwell model is in series. The Love coefficients can be written for any combination of jumps and oscillation damping. /273

Once the elastic problem is solved, the appropriate solution for the M and K-V models can be found by resubstitution of the dimensionless μ (and not $\bar{\mu}$), with the coefficients

$$\hat{\mu}_M = \frac{\mu\hat{D}}{\hat{D} + \tau^{-1}} \qquad \hat{\mu}_{K-V} = \mu(1 + \tau\hat{D})$$

where $r = \overline{n}/\overline{\mu}$ is the characteristic time and $\hat{D}$ the operator d/dt.

The Love operators $\hat{K} = K_f/(1+\overline{\mu})$, $\hat{K}' = -(1-L)/(1+\overline{\mu})$ are suitable. For the case of simple harmonic motion $e^{i\sigma t}$, the operator $\hat{D}$ becomes $i\sigma$ and the $\overline{\mu}$, $\hat{K}$, $\hat{K}'$ become the complex numbers μ, K, K'.

12.5. A Solution of Liouville's Approximation Equation

The Liouville equations are very greatly simplified by a "perturbation plan." The Earth's deformation is taken into account by various Love numbers.

12.5.1. Perturbations

The following method is suitable if the figure poles and the rotation are not very far from the reference pole.

$$\left.\begin{array}{lll} C_{11} = A + c_{11} & C_{22} = A + c_{22} & C_{33} = A + c_{33} \\ C_{12} = c_{12} & C_{13} = c_{13} & C_{23} = c_{23} \\ \omega_1 = \Omega m_1 & \omega_2 = \Omega m_2 & \omega_3 = \Omega(1+m_3) \end{array}\right| \quad (12.1)$$

where A, A, C are the moments of inertia with respect to the principal axes, Ω is the mean angular velocity of the earth, $\Omega = 0.729.10^{-4}$ rad/sid.s, C_{ij}/c, m_i and $g_i/\Omega c$ are small quantities the squares and products of which can be neglected. /274

Then the Liouville equations take the following simple form:

$$\frac{\dot{m}_1}{\sigma_r} + m_2 = \phi_2 \; , \quad \frac{\dot{m}_2}{\sigma_r} - m_1 = -\phi_1 \quad , \quad \dot{m}_3 = \dot{\phi}_3 \qquad (12.2)$$

where ϕ_i and σ_T are defined from the relations

$$\sigma_T = \frac{C-A}{A}\,\Omega \qquad (12.3)$$

$$\left.\begin{array}{l} \Omega^2 (C-A)\,\phi_1 = \Omega^2 c_{13} + \Omega\,\dot{c}_{23} + \Omega g_1 + \dot{g}_2 - L_2 \\ \Omega^2 (C-A)\,\phi_2 = \Omega^2 c_{23} - \Omega\,\dot{c}_{13} + \Omega g_2 - \dot{g}_1 + L_1 \\ \Omega^2 C \phi_3 = -\Omega^2 c_{33} - \Omega g_3 + \Omega \int_0^t L_3\, dt \end{array}\right| \qquad (12.4)$$

The left-hand side of (12.2) is determined by astronomical observations, and the right-hand side by geophysical observations. The dimensionless "excitation function" ϕ_i includes all the possible geophysical results of the motion of the Earth. The length of the day (l.o.d.) [sic].

In the third part of (12.2), m_1, m_2, ℓ are the direction cosines of the axis of rotation. On the complex plane, this will be

$$\bar{m} = m_1 + i\,m_2\ ,\quad \bar{\phi} = \phi_1 + i\phi_2 \Big\} \Longrightarrow\ i\,\frac{\dot{\bar{m}}}{\sigma_2} + \bar{m} = \bar{\phi} \tag{12.5}$$

12.5.2. Free Wobble

In the case of a free nutation of a perfectly rigid Earth, $\bar{\phi} = 0$ and the expression $(i\sigma_T t)$ is a solution of (12.5). The period $2\pi/\sigma_T$ is about 10 months. The role of the deformation of the Earth is to increase the period of free nutation by an essential fraction, about 40%. This result is not at all obvious and in fact it was not predicted before Chandler's discovery of the 14-month period in the change in latitudes. The qualitative explanation is as follows. For a perfectly rigid Earth, the frequency of the free nutation is proportional to the equinoctial swelling. For a deformable Earth, it depends only on that part of the equinocital swelling which does not adjust to the instantaneous axis of rotation. /275

Consider the "equation of excitation" which is due only to rotational deformation. The products of inertia arising from the rotational deformation are given by the relation

$$C_{ij} = \frac{k}{k_\ell}\,\frac{\omega_i\,\omega_j}{\Omega^2}\,(C-A)\ ,\quad i \neq j$$

already mentioned. In terms of perturbations noticed, these are

$$C_{13} = (C-A)\,\frac{k}{k_\ell}\,m_1 \qquad C_{23} = (C-A)\,\frac{k}{k_\ell}\,m_2 \tag{12.6}$$

When these expressions are substituted in (12.4), we obtain for the excitation equation

$$(\bar{\phi}) = \bar{\psi}_D - i\,\Omega^{-1}\dot{\bar{\psi}}_D \qquad \text{where}\quad \bar{\psi}_D = (k/k_\ell)\,\bar{m} \tag{12.7}$$

is a suitable note. The Eq. (12.5) becomes

$$\frac{i\dot{\bar{m}}}{\sigma_\tau} + \bar{m} = \bar{\psi}_D - i\Omega^{-1}\dot{\bar{\psi}}_D \simeq \bar{\psi}_D \qquad (12.8)$$

The approximation depends on $\sigma_\tau/\Omega = (C-A)A$, i.e. if it is a small number. The error is 0.1%. Hence (12.7) can be interpreted as that part of the excitation function $\bar{\phi}$ which is due to the rotational deformation. Equations (12.7) and (12.8) can be written according to the formula $i\dot{\bar{m}} + \sigma_0\bar{m} = 0$ which differs from the corresponding equation for a perfectly rigid Earth (with $\bar{\phi} = 0$), $i\dot{\bar{m}} + \sigma_\tau\bar{m} = 0$, at this frequency of the free nutation which was reduced from σ_2 to

$$\sigma_0 = \sigma_z \frac{k_f - k}{k_f} \qquad (12.9)$$

For an equivalent Earth, the value is

$$\frac{\sigma_0}{\sigma_z} = \frac{\mu}{1+\mu} = \frac{2,3}{3,3} \simeq 0.70$$

The principal axes of a body with moments of inertia and inertia products A, A, C, C_{12}, C_{13}, C_{23}, are inclined (approximately) by $C_{13}/(C-A)$, $C_{23}/(C-A)$ relative to the reference axis x_3. This results from (12.6) and (12.7). So $\bar{\psi}_\Delta$ describes the inclination of the principal axis of the rotationally (by rotation) deforming equinoctial swelling, the "axis of deformation." The positions where the $\bar{\psi}_\Delta$ axis intersect the surface are the poles of deformation. /276

The three models in the table below are instructive. In the case of a liquid Earth, the equinoctial swelling adjusts perfectly to the rotation, and there is no rotational stability and vibration $\sigma_0 = 0$. Only the part of the equinoctial swelling which remains fixed during wobble (≈70%) yields any stability. (σ_τ, σ_0 = corresponding frequencies, Tidal and Chandrel [sic],

$$Hz \quad rad/month \quad , \frac{2\pi}{10} \; , \frac{2\pi}{14})$$

12.5.3. Forced Wobble

Consider an oscillation due to any process following a path. We will first calculate the excitation function $\bar{\phi}(t)$ as if the Earth were perfectly rigid. The effect of the rotational

TABLE.

Model	Tidal effective Love numbers	Deformation axis	Nutation frequency
Perfectly rigid Earth	$K = 0$	$\bar{\psi}_D = 0$	$\sigma_0 = \sigma_E = 1$ cycle/month
Liquid Earth	$K = K_f$	$\bar{\psi}_D = \bar{m}$	$\sigma_0 = 0$
Real Earth	$K = 0{,}29$	$\bar{\psi}_D = 0{,}90\bar{m}$	$\sigma_0 = \frac{\sigma_E}{1{,}4} = 1$ cycle/month

deformation is to produce an additional excitation:

$$\bar{\psi}_D = \left(\frac{K}{K_f}\right)\bar{m} \tag{12.10}$$

which must be taken into account during the prescribed excitation. If the initial path of the processes does not load the Earth (e.g., winds), then the total excitation function consists of two parts, i.e. $\bar{\phi} = \bar{\psi} + \bar{\psi}_D$. But if the Earth is not loaded,

$$\longrightarrow \bar{\phi} = \bar{\psi} + \bar{\psi}_D + \bar{\psi}_L \tag{12.11}$$

where $\bar{\psi}_L = K'\psi$ is an additional excitation arising from the deformation loading. The effect of deformation due to rotation is then to cause a larger excitation (and wobble) than the one obtained for a perfectly rigid Earth. The effect of the deformation is to decrease the total excitation (K' is negative). We can define, for the interpretation, the deformation due to rotation as a positive reaction and the deformation due to loading as a negative reaction. This is suitable for the definition of a "modified excitation": /277

$$\hat{\psi} = K_{ta_\lambda} \cdot \bar{\psi} \tag{12.12}$$

where K_{ta_λ} is a "transfer function," equal to

$$K_{TAJ} = \frac{K_f}{K_f - K} \qquad \text{and} \qquad K_{TAJ.} = (1+k')\frac{K_f}{K_f - K}$$

depending on whether or not the process loads or does not load the Earth. Equation (12.5) can be written in the following equivalent form

$$\dot{\tilde{m}} = i\sigma_\tau(\tilde{m} - \tilde{\phi}), \quad \dot{\tilde{m}} = i\sigma_0(\tilde{m} - \tilde{\psi}) \tag{12.13}$$

The two forms differ with respect to the frequency and excitation functions. The total excitation $\bar{\phi}$ includes the deformation $\bar{\psi}_D$ due to rotation, and when it is combined with $\bar{m}$, then σ_τ becomes σ_0 and ϕ becomes ψ (term (12.13)).

12.5.4. "Transfer Function"

The transfer function K for the equivalent Earth is obtained from the definitions of K, K' where

$$K = \frac{K_f}{1+\mu}, \quad K' = -\frac{1}{1+\mu}$$

The values are $K_{TAJ.} = \frac{1+\mu}{\mu} = 1{,}43$, $K_{TAJ.} = 1$

depending on whether the process is loading the Earth or not. In the following case, the increase from deformation due to rotation cancels the decrease from deformation due to loading, and the resulting wobble is the same as if the Earth were perfectly rigid. At first sight, the result seems surprising. /278
But the deformation due to loading contributes only to the products of inertia, and these are spherical harmonics of β' degree and of the same type as the deformation due to rotation.

The value of the transfer function K (wobble): no load = 1.43, load = 1.00. These results are compared to the astronomical observations by virtue of the relation

$$\dot{\tilde{m}} = i\sigma_0(\tilde{m} - \tilde{\psi}) \tag{12.14}$$

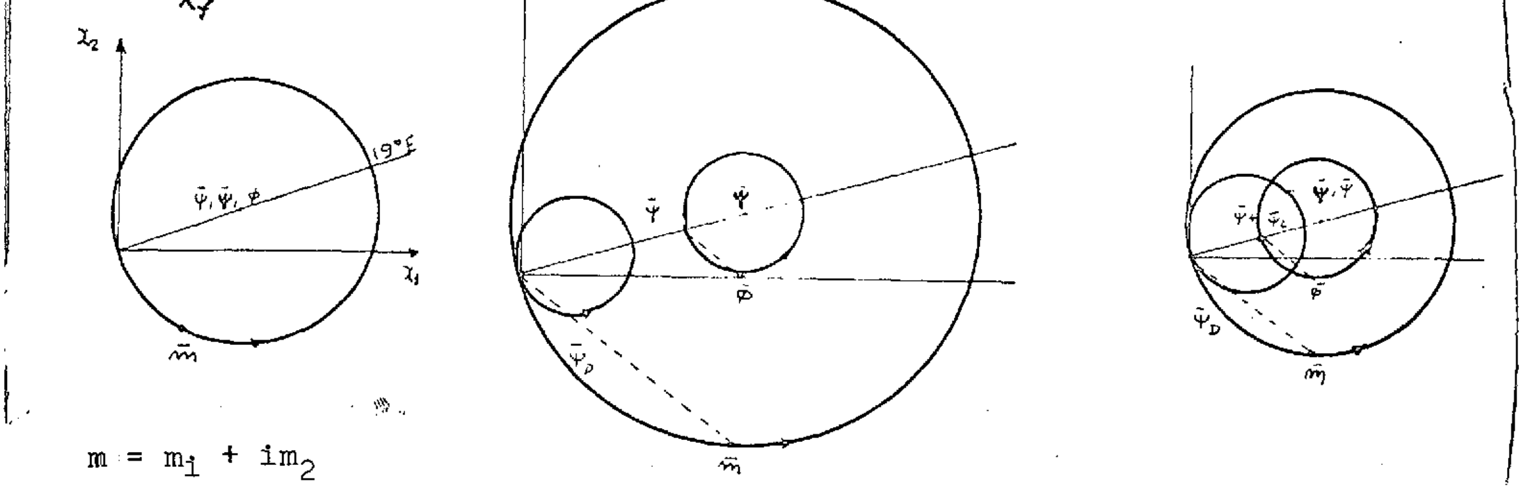
However, the above values must be used carefully. These values are based on the stiffness of the tidal effect as it is obtained from the wobble. At high frequencies, the values fail, because the correspondence of the oceans and the nucleus must be independent of the frequency. At very low frequencies the nonelastic deformation of the shell may play an important role, and the transfer function is then a pure number.

12.5.5. A Geometric Representation

We can take the vector sum of the effects of rotational and loading deformations. Initially, the excitation and the poles of rotation are given at the origin of the coordinates. Hence, $\bar{\phi} = 0$, $\bar{m} = 0$. At time 0, the excitation pole is suddenly displaced (in the direction 19° East) due to some special event.

In the diagram below, the Earth is assumed to be perfectly rigid and the rotational pole m is taken as rotating around the excited pole ($\bar{\psi} = \bar{\phi}$) as shown. In the central diagram, the case of deformation due to an excitation which does not load the Earth is shown ($\psi_L = 0$). The new feature is that the equinoctial swelling adjusts to the perturbed position of the pole of rotation. The figure of the swelling tends to come along a line perpendicular to the axis of rotation, but in exposing the elastic only 1-3 successful [sic] is necessary: /279

$$\tilde{\psi}_D = \frac{k}{k_f}\,\bar{m} = 0{,}30\,\bar{m}$$



$m = m_1 + im_2$

m_1, m_2 = direction cosines

The coordinates x_1, x_2 are given by the undisturbed pole in the direction of Greenwich and 90° East of Greenwich. The disturbed excited pole $\bar{\psi}$ is displaced 19° East. The partial results of the pole of rotation $\bar{m}$ are given for a perfectly rigid Earth above, for a deformed Earth, in the case of no load, due to excitation (middle), and for a loaded and excited Earth, at the bottom. The initial specific excitation $\bar{\psi}$ is the same in all three cases. The figures show the positions of the pole of deformation $\bar{\psi}_D$, of loading $\bar{\psi}_L$, of total excitation $\bar{\phi}$ and of modified excitation $\bar{\psi}$.

The total excitation of the pole $\bar{\phi} = \bar{\psi} + \bar{\psi}_D$ consists of the part $\bar{\psi}$ which was calculated on the basis of the assumption that the Earth is a perfectly rigid body, increased by the additional part $\bar{\psi}_D$ arising from the deformation. The pole of rotation m rotates around the average position $\bar{\psi}$ of the excited pole with a /280
radius magnified by a factor $K_f/(K_f - K) = 1.43$ compared to the case of perfect rigidity, but the distinction $\bar{m} - \bar{\phi}$ between the pole of rotation and the instantaneous excited pole is the same as in the case of perfect rigidity. The fact that pole $\bar{\psi}$ of a modified excitation is also the average position of the instantaneous pole can be proved as follows. Assume that $\bar{\psi}$ is at the center of the concentric circle described by $\bar{m}$ (of radius R) and by $\bar{\phi}$. We can write

$$\bar{m} = \bar{\psi} + \bar{\alpha} R, \qquad \bar{\phi} = \bar{\psi} + \bar{\alpha}\left(\frac{K}{K_f}\right) R,$$

where $\bar{\alpha}$ is the unit vector. We form $(K/K_f)\bar{m} - \bar{\phi}$ and take account of the relation

$$\bar{\psi}_D = K/K_f\, \bar{m} \Rightarrow \quad \Rightarrow \bar{\psi}_D - \bar{\phi} = \bar{\psi}\left(\frac{K}{K_f} - 1\right).$$

If we also use the relations of forced oscillation, we obtain $\bar{\psi} = K\bar{\psi}$ which is in accordance with the definition of $\bar{\psi}$.

The velocity of wandering of the pole of rotation α is proportional to $\bar{m} - \bar{\phi}$ (see (12.13)), and therefore invariable with respect to deformation, but the period of a full rotation increases from 10 to 14 months as the radius increases respectively. The situation is more complicated if the excited pole loads the Earth (right figure). The excitation $\bar{\psi} + \bar{\psi}_L$ for deformation due to loading is smaller by δ than the excitation $\bar{\psi}$ due to the initially described surface loading. This decrease is canceled by the increase due to rotational deformation, so that finally the radius of the circle described by the pole of rotation is the same as for a perfectly rigid Earth. As always, the velocity of wandering (rotation of the pole) is proportional to $(\bar{m} - \bar{\phi})$. This is reduced by a factor of 1.4 in comparison to the perfectly rigid Earth, and therefore the period of rotation increases from 10 to 14 months. /281

12.5.6. Excitation Function

Equations (12.4) give the total excitation which includes deformations due to loading and rotation. In all practical problems, we are obliged to estimate the excitation ϕ_i as if the Earth were perfectly rigid, and to take into account the secondary deformation by means of the transfer function. In this context, (12.4) can be used as written, substituting ϕ_i for ϕ_j.

Equations (12.4) are well fitted for computing the excitation function, whenever it changes in angular momentum of motion, and it is independent of the inertia product changes. This usually arises whenever one or the other vanishes, for example, a power-emitting wheel fixed on the ground and rotating with variable value of angular velocity gives different values of relative angular momentum but not of the moment of inertia. In the case of melting ice, the angular momentum of the flowing water is negligible, but it changes the moment of inertia.

Equations (12.4) are not suitable if we wish to separate explicitly the results due to change in material disturbance from those due to relative velocity. The reason is that $\dot{C}_{ij}$ and g_i are both included in the relative velocity. Also they are of the same order. Equations of a separate type were used by Munk and Groves (1952) in calculating the annual wobble due to wind and ocean currents. Equations (12.4) can be written:

/282

$$\Omega^2(C-A)\,\tilde{\phi} = \int_v \Delta\rho\, \tilde{F}(\text{matter})\,dv + \int_v \rho\, \tilde{F}(\text{motion})\,dv + \tilde{F}\,(\text{moment}) \qquad (12.15)$$

$$\Omega^2 C\,\phi_3 = \int_v \Delta\rho\, F_3(\text{matter})\,dv + \int_v \rho\, F_3(\text{motion})\,dv + F_3\,(\text{moment})$$

where $\Delta\rho(x_i,t)$ is the density related to the excitation equations $\Phi_K(t)$ and where:

matter:

$$F_1 = -\Omega^2 x_1 x_3\,, \quad F_2 = -\Omega^2 x_2 x_3\,, \quad F_3 = -\Omega^2(x_1^2 + x_2^2)$$

$$: F_1 = -2\varrho x_3 u_2 + x_3 \dot{u}_1 - x_1 \dot{u}_3 \quad , \quad F_2 = 2\varrho x_3 u_1 + x_3 \dot{u}_2 - x_2 \dot{u}_3$$

motion: $$F_3 = \varrho(-x_1 u_2 + x_2 u_1)$$

moment: $$F_1 = -L_2 \quad , \quad F_2 = L_1 \quad , \quad F_3 = \varrho \int_0^t L_3 dt$$

are functions depending on the material disturbance, the relative motion (velocity and acceleration), and the moment. Often spherical coordinates are suitable. Consider that K_λ, K_θ, K_r represent the East, South and upward components of the velocity and

$$dv = r^2 \sin\vartheta \, dr \, d\vartheta \, d\lambda$$

the differential volume. Then we will have

matter: $$F_1 = -r^2 \varrho^2 \sin\vartheta \cos\vartheta \cos\lambda \, , \quad F_2 = -r^2 \varrho^2 \sin\vartheta \cos\vartheta \sin\lambda \, , \quad F_3 = -r^2 \varrho^2 \sin^2\vartheta$$

motion:

$$F_1 = -2\varrho r \cos\vartheta \left(u_\lambda \cos\lambda + u_\vartheta \cos\vartheta \sin\lambda + u_r \sin\vartheta \sin\lambda\right) + r\left(-\dot{u}_\lambda \cos\vartheta \sin\lambda + \dot{u}_\vartheta \cos\lambda\right)$$

$$F_2 = 2\varrho r \cos\vartheta \left(-u_\lambda \sin\lambda + u_\vartheta \cos\vartheta \cos\lambda + u_r \sin\vartheta \cos\lambda\right) + r\left(\dot{u}_\lambda \cos\vartheta \cos\lambda + \dot{u}_\vartheta \sin\lambda\right)$$

$$F_3 = -\varrho r \sin\vartheta \, u_\lambda$$

The moment can be written as the sum of two terms:

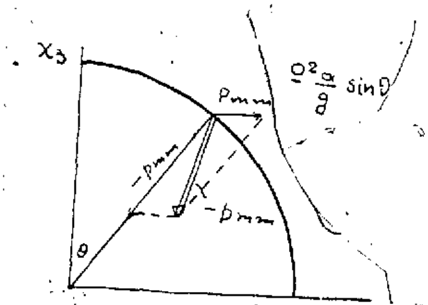
$$\rightarrow L_i = \int_v \rho \, e_{ijk} \, x_j f_k \, dv + \int_v \varepsilon_{ijk} \, x_j \, p_{km} \, n_m \, dS$$

The first part is due to the body's force f_K, for example the gravitational attraction of the equinoctial swelling. The β' term is due to the surface tension D_{Km} in direction K on a surface element normal to n_m. As a special case, we consider a smoothed geoid, the surface of which is normal at each point to the vector of gravity passing through the point under consideration. The radial components of the surface tension do not yield a moment. The nonradial components are: /283

$$p_{1m} \simeq p_{\vartheta m} \cos\vartheta \cos\lambda - p_{\lambda m} \sin\lambda + p_{mm} (\Omega^2 a/g) \sin\vartheta \cos\lambda$$

$$p_{2m} \simeq p_{\vartheta m} \cos\vartheta \sin\lambda + p_{\lambda m} \cos\lambda + p_{mm} (\Omega^2 a/g) \sin\vartheta \sin\lambda$$

$$p_{3m} \simeq - p_{\vartheta m} \sin\vartheta$$



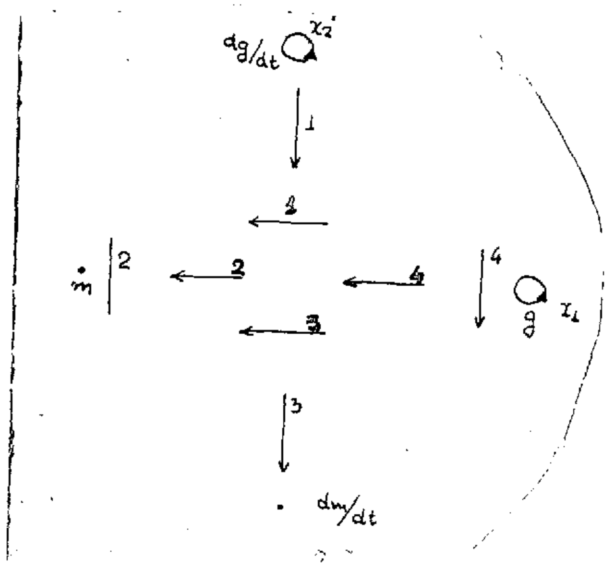
where p_{mm} is the stress component along the normal on the geoid. (The summation convention does not apply here.) Hence:

$$L_1 = a\int_s \left[-p_{\vartheta m} \sin\lambda - p_{\lambda m} \cos\vartheta \cos\lambda - p_{mm} \Omega^2 \frac{a}{g} \sin\vartheta \cos\vartheta \sin\lambda\right] ds$$

$$L_2 = a\int_s \left[p_{\vartheta m} \cos\lambda - p_{\lambda m} \cos\vartheta \sin\lambda + p_{mm} \Omega^2 \frac{a}{g} \sin\vartheta \cos\vartheta \cos\lambda\right] ds$$

$$L_3 = a\int_s \left[p_{\lambda m} \sin\vartheta\right] ds$$

A graphic summary of the effect of the various events on the excited pole is given in the figure below. It is understood that a local left-handed rotation has a similar result to a local failure of mass. Both cyclones are in the Northern hemisphere.



1, 2, 3, 4 = toward North, East, South, West motion.

1, 2, 3, 4 = toward East, South, West, North stress.

Disturbances in the moment are due to a mass, and at 180° this result has a positive value of increase dm/dt at 270° East. The effect of the relative angular momentum g on a horizontal vortex and the time change dg/dt are shown. /284

12.6. Eternal Polar Motion

We have already mentioned that eternal polar motion of the pole consists of a progressive component and a periodic equilibrium (wobble). It is proved that the periodic wobble takes place with a period of 24 years.

The total deformation excitation $\bar{\psi}$ due to the displacement of the Earth will be proportional to Kd where K is the Love number and d the distance of the pole of rotation from the axis of the figure. Furthermore, if change in τ, the free nutational period, is caused only by the displacement of the Earth, will be a simple function of K. Nevertheless, τ changes with d, and therefore $\bar{\psi}$ may include a function of d^2. The value of d is given by the free nutational term m_o, of the form $(i\sigma_o t)$ plus the forced nutational terms m, of the form $(i\sigma t)$ and n, of the form $(-i\sigma t)$. Then it can be shown that this leads to mean excitations varying as $\cos(\sigma_o \pm \sigma)t/2$ and that these are caused by an excitation which changes as $\cos(\sigma - \sigma_o)t/4$. The frequencies $(\sigma - \sigma_o)/2$ and $(\sigma - \sigma_o)/4$ correspond to the observed wobble periods of about 12 and 24 years.

The computed meridian of the wobble is in satisfactory agreement with the observed meridian λ_L. The angle between λ_L and λ_P, the meridian of the "progressive" motion, is given by $\lambda = \cos t^{-1}((K/K_f)\cos t\theta)$ where K_f is the value of K for hydrostatic equilibrium and θ is the angle between λ_P and the principal axis of epochic excitation. The amplitudes of the wobble and the apparent fluctuations in progressive motion change with σ_o in an expected way. When the natural frequency σ_o comes close to the forced frequency σ, the excitations and amplitudes increase, and the mean pole advances toward the direction of the progressive motion. /285

Changes in the length of the day T show as a result a value of K apparently changing with d. The observed value of T and the amplitude of polar motion from 1955 show similarly a 6 year change proportional to $|\cos(\sigma - \sigma_o)t/2|$.

The value of K and that of τ depend inversely on the mean excitation. When wobble excitation increases, the values of τ decrease in a different way. Moreover, λ increases more when K increases less. The 6-year excitation which is a consequence of the free and forced wobble, shows a similar result. When the consequences are greater, the excitations caused are smaller and the values of τ are smaller.

Exactly these relations show that in periodic excitations, nutations are connected with the wobbles and with the change in the length of the day.

12.7. Small Period Changes

The small period changes and the totally irregular changes have not been investigated. These changes are caused, on the one hand, by epochic periodic phenomena, or on the other hand, by phenomena which take place suddenly. Such phenomena are wind, ocean currents, earthquakes, and especially tectonic phenomena, sudden volcanic explosions, accidental meteorite falls, etc. It was found that if all the cars in America were driven from Alaska to Mexico, the moment of inertia of the Earth would change by $1/10^4$. That is, there are known and unknown phenomena causing changes. A common feature of them is that the change in polar position which they cause is very small. Attempts have been made to investigate all these phenomena, and mathematical formulations have been found for some of them, and their investigation is proceeding rapidly. Obviously, after all of them have been investigated, we will be able to attribute to each of them the corresponding amount of total change in polar position. In any case, the consideration of all phenomena seems difficult enough. /286

With this, the above investigation closes, and it is considered that it touched the main problems, theories and methods, unraveling the knot of the major part of knowledge about polar motion to future works.

REFERENCES

1. Yumi, Shigeru, Annual Report of the International Polar Motion Service of the Years 1962, 1963, 1964, 1965, 1966, 1967, Mizusawa.

2. Lissimachos, Mavridis, Astronomia [Astronomy], Thessaloniki, 1969.

3. Munk and MacDonald, The Rotation of the Earth.

4. Nitsiotis, G., Eisagōgē eis tēn statikēn [Introduction to Statics], Thessaloniki, 1965.

5. Hatzidimitriou, I., Theōrētikē mehanikē [Theoretical Mechanics], Thessaloniki, 1970.

6. Continental Drift, Secular Motion of the Pole, and Rotation of the Earth, 32nd Symposium.

7. Brikas, Mavrikos, Mathēmata tanustikou logismou [A Course in Tensor Calculus], Athens, 1958.

8. Argyrakos, Geōdaitikē astronomia [Geodetic Astronomy], Athens, 1969.

9. Monthly Notes of the International Polar Service for the Years 1962-1965.

10. Nitsiotis, G., Antohē tōn hylikōn [Strength of Materials], Thessaloniki, 1960.

11. Arfken, G., Mathematics for Physicists, U.S.A.

12. Fedorov, Nutation and Forced Motion of the Earth's Pole, Oxford, 1963.

STANDARD TITLE PAGE

1. Report No. NASA TT F-15,066

2. Government Accession No.

3. Recipient's Catalog No.

4. Title and Subtitle

A STUDY OF POLAR MOTION

5. Report Date October, 1973

6. Performing Organization Code

7. Author(s)

A. Terpos
University of Thessaloniki Polytechnic School

8. Performing Organization Report No.

10. Work Unit No.

9. Performing Organization Name and Address

Leo Kanner Associates, P.O. Box 5187
Redwood City, California 94063

11. Contract or Grant No. NASw-2481

13. Type of Report and Period Covered

Translation

12. Sponsoring Agency Name and Address

National Aeronautics and Space Administration, Washington, D.C. 20546

14. Sponsoring Agency Code

15. Supplementary Notes

Translation of Peri tēs kinēseos tou polou tēs gēs, Dissertation for the Master's Degree, Division of Rural and Surveying Engineering, University of Thessaloniki Polytechnic School, 1971, 287 pp.

16. Abstract

Field measurements relative to polar motion and a model of the Earth that will give the expected results are presented. The changes in the axis of rotation as a result of precession, nutation, and wobble are discussed, and their causes are investigated. Methods of observation and the establishment of a general coordinate system as a master reference for latitude measurements are described. Initial data from five ILS stations are included, and the methods used by the Central Bureau in analyzing the data are outlined. Possible errors in azimuth and inclination data are treated, along with correction methods, emphasizing the chain method. The model of the Earth is based on the Earth's ability to react to the forces of nature; the Earth is considered as essentially stable and solid. Theories treating the Earth as an elastic body are also discussed. Tensor analysis is employed throughout.

N73-18320

17. Key Words (Selected by Author(s))

18. Distribution Statement

Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	243	